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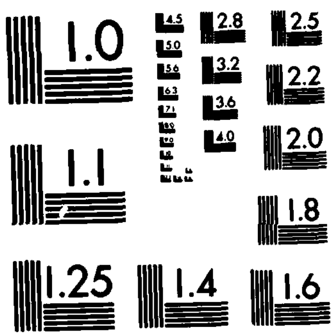

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Analytical development for the generation of three-dimensional curvilinear coordinates and the consequent computational code have been completed. The computational code has all the necessary subroutines for use in the generation of curvilinear coordinates between a single inner and an outer given body. Work on multielement bodies is continuing.

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The Generation of Three-Dimensional Body-Fitted Coordinate Systems For Viscous Flow Problems

By

Z. U. A. Warsi[*], C. W. Mastin[**] and J. F. Thompson[***]

Abstract

The proposed set of equations (refer to the enclosed papers for detail) which generate a series of surfaces between a given inner and outer body, have been programed on CRAY-1. An extensive program testing has been carried out to make the program usable for general body shapes. For example, two methods have been developed to establish the correspondence between the points of the inner and outer surfaces, a method has been developed to find the first partial derivative of x, y, z with respect to the coordinate along the surface. All these methods are based on sound mathematical basis and have not been chosen arbitrarily. Work has been started on complicated multibody configurations, such as the wing-body combination. Here the interfacing of coordinates having sufficient derivative smoothness is the most important problem.

In the period of this report, a thorough analysis on all sorts of mathematical models for coordinate generation has been completed. This analysis uncovers those differential relations which must invariably be satisfied by the metric coefficients no matter which method is used

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to generate the coordinates. The final computational code to be developed in the current year of effort is expected to reflect all these achievements for practical utilization.

Previously reported examples of surface grid construction have dealt with simple quadratic surfaces. To test the versitility of the method, our program has been modified to accept the surface data generated by Craidon [1][†] Aside from the more complex computational region, which gave rise to additional coordinate singularities, the grid generation procedure was unchanged. That is, the grid was generated using cubic splines with an elliptic system used for smoothing. Several coordinate surfaces for the region about a spline generated wing-fuselage configuration have been plotted. A plot of that configuration is given in Figure 1(a). Figure 1(b) illustrates the continuation of coordinate lines from the body to the outer boundary. Views of coordinate surfaces from the upstream direction are also included. Figure 1(c) is a surface surrounding the aircraft, and Figure 1(d) is a surface intersecting the trailing edge of the wing.

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[†]C. B. Craidon, "A Computer Program for Fitting Smooth Surfaces to an Aircraft Configuration and Other Three-Dimensional Geometries," NASA TM X-3206, 1975.

<u>Papers Written Under The Contract</u>

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2. "Numerical Generation of Three-Dimensional Coordinates Between Bodies of Arbitrary Shapes", Z. U. A. Warsi, and J. P. Ziebarth, <u>Numerical Grid Generation</u>, Elsevier Scientific Publishing Co. To be published by Sept. (1982).

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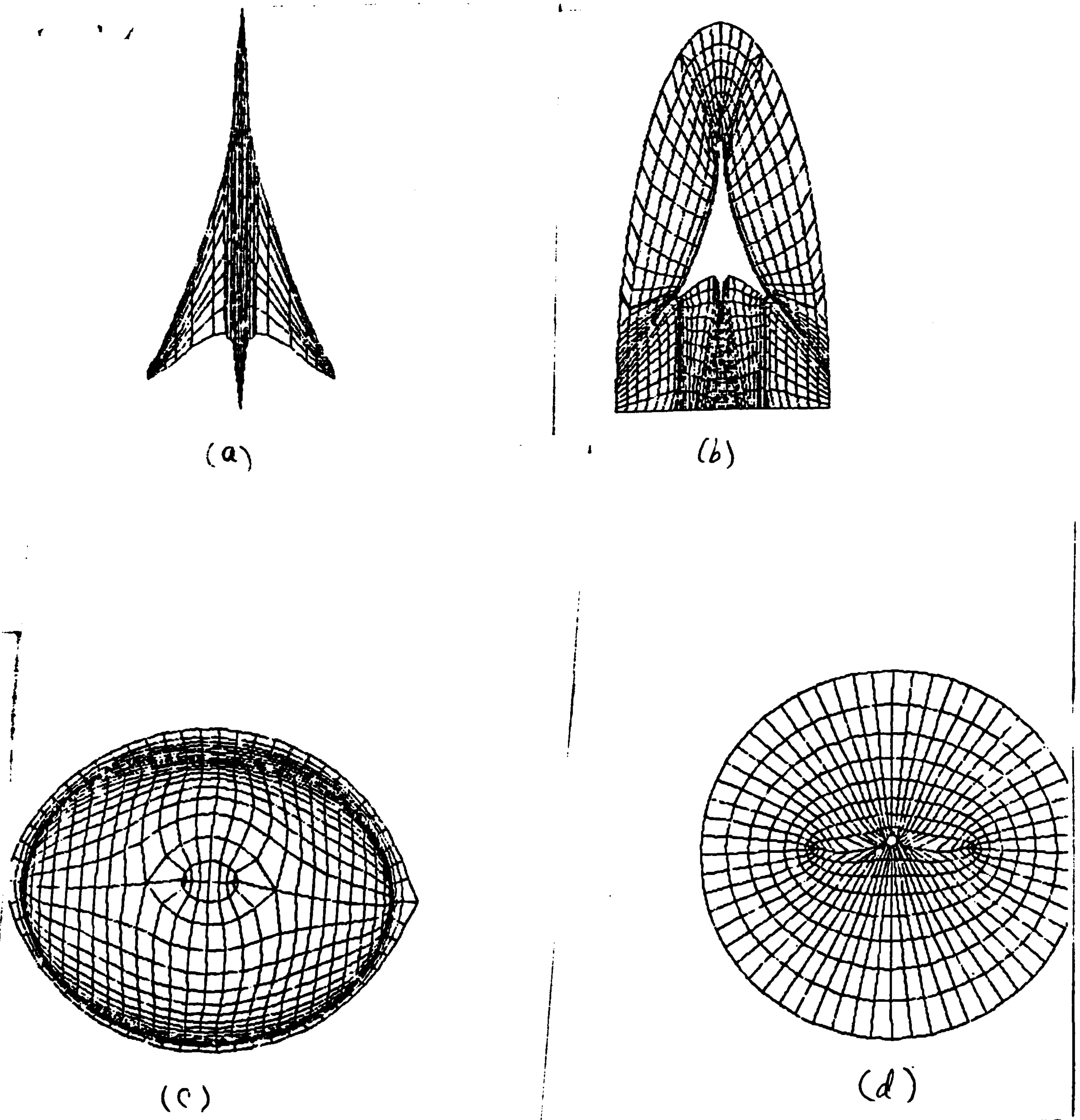

Figure 1. Coordinate Surfaces

BASIC DIFFERENTIAL MODELS FOR COORDINATE GENERATION

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§1. Introduction

§2. Notation and Basic Formulas

§3. Generating Differential Equations Based on Gauss Equations

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§1. INTRODUCTION

This paper examines in detail the analytical aspects of three distinct methods of coordinate generation based on partial differential equations, in either two or three dimensions. The first method is based on the Gauss equations of a surface under the constraint of the Beltrami's second order equations. These equations have been structured in such a way that an automatic connection is established between the succeeding generated surfaces. The second method is a re-examination of those equations which are based on the inhomogeneous Laplace equations. This analysis reveals a new form for the terms which play a role in the concentration of coordinate lines and in the adaptive coordinate system generation. The third method pertains to a set of equations in the metric coefficients which is obtained by setting the Riemann's curvature tensor to zero.

The problem of generating spatial coordinates by numerical methods is a problem of much interest in practically all branches of engineering and physics. At present a number of techniques are under active development for the generation of two and three-dimensional coordinates in the regions between two or a number of arbitrary shaped bodies. Among these efforts two easily discernable groups can be formed, (i) the methods based on elliptic PDE's, and (ii) algebraic methods. In the first group, a set of inhomogeneous Laplace equations is taken as the basic generating system. These equations are then inverted and solved for the Cartesian coordinates.

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‡Professor

Some very useful results based on this line of approach started with the work of Winslow[1], have been obtained by Thompson, et al.[2] (TTM method), Steger, et al.[3], Yu[4], Graves[5], and Thomas[6]. For an extensive bibliography refer to Thompson, et al.[7] In the second group of methods, the grid points in space are generated by interpolating and blending functions starting from the given boundary data. This line of approach has been followed by Eiseman[8], Smith, et al.[9], Erickson[10], and others.

In this paper we consider only the analytical aspects of the differential equation's approach to coordinate generation. The main effort here is to present only those results which are of permanent interest to the workers in the field of coordinate generation. The proposed equations in any one of the groups have not been arbitrarily selected to generate some sort of coordinates. These equations are in fact those which every numerically or analytically generated coordinates must satisfy. The reader will find that some large portions of sections 3 and 5 have new results and are based on the work by Warsi[11,12]. In sections 3 and 5 a number of exact solutions have been obtained which can be used to provide a testing ground for different numerical schemes.

§2. NOTATION AND BASIC FORMULAS

In this paper any general curvilinear coordinate system will be denoted by a superscript index notation, such as x^i. However, when an expression has been expanded out in full and there is no need for an index notation then we shall use the symbols

$$x^1 = \xi \ , \ x^2 = \eta \ , \ x^3 = \zeta \ .$$

The rectangular Cartesian coordinates (x, y, z) which determine the position vector $\underset{\sim}{r}$, i.e.,

$$\underset{\sim}{r} = \underset{\sim}{r}(x,y,z)$$

will be denoted by the subscripted variable x_i, where $x_1 = x$, $x_2 = y$, $x_3 = z$.

Two similar indices, one appearing as a subscript and the other as a superscript will always imply summation over the range of index values; e.g., $A_{ij}B^i = A_{1j}B^1 + A_{2j}B^2 + A_{3j}B^3$.

In an Euclidean space (E^2 or E^3), the covariant base vectors $\underset{\sim}{a}_i$ are given by

$$\underset{\sim}{a}_i = \frac{\partial \underset{\sim}{r}}{\partial x^i} \ , \tag{1a}$$

so that

$$\underset{\sim}{a}_1 = \underset{\sim}{r}_\xi \ , \ \underset{\sim}{a}_2 = \underset{\sim}{r}_\eta \ , \ \underset{\sim}{a}_3 = \underset{\sim}{r}_\zeta \ , \tag{1b}$$

where a variable subscript will denote a partial derivative. Using the Riemannian metric, the formula for the length element ds is given by

$$(ds)^2 = g_{ij} dx^i dx^j \ ,$$

where, because of the Euclidean nature of the space the metric coefficients are given by

$$g_{ij} = \underset{\sim}{a}_i \cdot \underset{\sim}{a}_j = \frac{\partial \underset{\sim}{r}}{\partial x^i} \cdot \frac{\partial \underset{\sim}{r}}{\partial x^j} \ . \tag{2a}$$

The coefficients $g_{ij} = g_{ji}$ are the covariant components of the metric tensor. The contravariant components g^{ij} are related with g_{ij} through the equation

$$g^{ij} g_{ik} = \delta^j_k \tag{2b}$$

where δ^j_k (the Kronecker deltas) are the mixed components of the metric tensor. Using (2b) we define the contravariant base vectors as

$$\underset{\sim}{a}^i = g^{ij} \underset{\sim}{a}_j \ . \tag{2c}$$

The quantities g and $\hat{g}$ defined as

$$g = \det(g_{ij}) \ , \tag{3a}$$

$$\hat{g} = \det(g^{ij}) \ , \tag{3b}$$

are related as

$$g\hat{g} = 1 \ . \tag{4}$$

For a three-dimensional space

$$g = g_{11}g_{22}g_{33} + 2g_{12}g_{13}g_{23} - (g_{23})^2 g_{11} - (g_{13})^2 g_{22} - (g_{12})^2 g_{33} . \tag{5}$$

Introducing the quantities,

$$
\begin{aligned}
G_1 &= g_{22}g_{33} - (g_{23})^2 , \\
G_2 &= g_{11}g_{33} - (g_{13})^2 , \\
G_3 &= g_{11}g_{22} - (g_{12})^2 , \\
G_4 &= g_{13}g_{23} - g_{12}g_{33} , \\
G_5 &= g_{12}g_{23} - g_{13}g_{22} , \\
G_6 &= g_{12}g_{13} - g_{23}g_{11} ,
\end{aligned} \tag{6}
$$

we have, on solving Eqs. (2b),

$$g^{11} = G_1/g \ , \ g^{22} = G_2/g \ , \ g^{33} = G_3/g \ , \tag{7a}$$

$$g^{12} = G_4/g \ , \ g^{13} = G_5/g \ , \ g^{23} = G_6/g \ . \tag{7b}$$

The space Christoffel symbols of the first and second kind respectively are given by

$$[ij,k] = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right) , \tag{8a}$$

$$\Gamma^{\ell}_{ij} = g^{k\ell}[ij,k] \ . \tag{8b}$$

Using (8b), we have

$$\frac{\partial \underset{\sim}{a}_i}{\partial x^j} = \Gamma^{\ell}_{ij}\, \underset{\sim}{a}_{\ell} \ . \tag{8c}$$

In the case of a two-dimensional surface embedded in a three-dimensional space, we shall use the Greek indices α, β, etc. (with the exception of ν) with the stipulation that they assume only two values. Thus the surface Christoffel symbols of the first and second kind are respectively given by

$$[\alpha\beta,\delta] = \frac{1}{2}\left(\frac{\partial g_{\alpha\delta}}{\partial x^{\beta}} + \frac{\partial g_{\beta\delta}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\delta}}\right) , \tag{9a}$$

$$T^{\sigma}_{\alpha\beta} = g^{\delta\sigma}[\alpha\beta\ \delta] , \tag{9b}$$

where for the purpose of clarification we have used the symbol Υ (upsilon) to denote the surface Christoffel symbols of the second kind in (9b) and not by Γ as in (8b).

In the process of formulation of a 3D coordinate generation problem, it is helpful to imagine the coordinates of a point in space as the intersection of three distinct surfaces on each of which one coordinate is held fixed. Using the convention of a right-handed coordinate system x^1, x^2, x^3 or ξ, η, ζ, we introduce the notation $\Sigma^{(\nu)}$ as a surface on which the coordinate x^ν = const., such that

$\nu = 1$ implies that (x^2, x^3) are in the surface,

$\nu = 2$ implies that (x^3, x^1) are in the surface,

$\nu = 3$ implies that (x^1, x^2) are in the surface.

Thus, the unit normal vector on the surface $\Sigma^{(\nu)}$ is

$$\underset{\sim}{n}^{(\nu)} = (\underset{\sim}{r}_\alpha \times \underset{\sim}{r}_\beta)/|\underset{\sim}{r}_\alpha \times \underset{\sim}{r}_\beta| , \tag{10}$$

where

$$\begin{aligned} \nu = 1 &: \alpha = 2 , \beta = 3 \quad (\text{surface } x^1 = \xi = \text{const.}) , \\ \nu = 2 &: \alpha = 3 , \beta = 1 \quad (\text{surface } x^2 = \eta = \text{const.}) , \\ \nu = 3 &: \alpha = 1 , \beta = 2 \quad (\text{surface } x^3 = \zeta = \text{const.}) . \end{aligned} \tag{11}$$

All other quantities and formulas which appear in the rest of the paper have been defined where they first appear. Refer also to Warsi[12] and Eisenhart[13].

§3. GENERATING DIFFERENTIAL EQUATIONS BASED ON GAUSS EQUATIONS

In this section our aim is to develop a method[11] for the generation of 3D coordinates wherein a series of surfaces are generated on each of which two previously designated coordinates vary while the third coordinate remains fixed. This method must also be structured in such a way that the variation of the third coordinate from one generated surface to the next is fully reflected in the system of generating equations. With this aim, we start from the equations of Gauss[13,14] which for a surface x^ν = const., are given by

$$\underset{\sim}{r}_{\alpha\beta} = T^\delta_{\alpha\beta}\, \underset{\sim}{r}_\delta + b_{\alpha\beta}\, \underset{\sim}{n}^{(\nu)} , \tag{12}$$

where the variations of α, β and the range of δ with ν follows the scheme in (11). The quantities $b_{\alpha\beta}$ are the coefficients of the second fundamental form of the surface. Since on the surface x^ν = const., the vector $\underset{\sim}{n}^{(\nu)}$ is orthogonal to the surface vectors $\underset{\sim}{r}_\delta$, hence

$$b_{\alpha\beta} = \underset{\sim}{n}^{(\nu)} \cdot \underset{\sim}{r}_{\alpha\beta} . \tag{13}$$

To fix ideas, we envisage a surface which is formed of the coordinate lines ξ, η and on which ζ = const. Dropping the index ν, Eq. (12) yields the three equations

$$\underset{\sim}{r}_{\xi\xi} = T^\delta_{11}\underset{\sim}{r}_\delta + S\underset{\sim}{n} , \tag{14a}$$

$$\underset{\sim}{r}_{\xi\eta} = T^\delta_{12}\underset{\sim}{r}_\delta + T\underset{\sim}{n} \tag{14b}$$

$$\underset{\sim}{r}_{\eta\eta} = T^\delta_{22}\underset{\sim}{r}_\delta + U\underset{\sim}{n} \tag{14c}$$

where the index δ now varies from 1 to 2, and

$$S = b_{11} , \quad T = b_{12} , \quad U = b_{22} . \tag{15}$$

Here $\underset{\sim}{n}$ is orthogonal to both $\underset{\sim}{r}_\xi$ and $\underset{\sim}{r}_\eta$, and the coefficients of the first fundamental form of the surface are g_{11}, g_{12}, and g_{22}; each evaluated at ζ = const. Obviously

$$g_{11} = x^2_\xi+y^2_\xi+z^2_\xi , \quad g_{12} = x_\xi x_\eta+y_\xi y_\eta+z_\xi z_\eta , \quad g_{22} = x^2_\eta+y^2_\eta+z^2_\eta . \tag{16}$$

If Eqs. (14) are considered as the first order partial differential equations in $\underset{\sim}{r}_\xi$ and $\underset{\sim}{r}_\eta$, then we must also consider the Weingarten equations

$$\underset{\sim}{n}_\xi = -b_{1\beta}g^{\beta\gamma}\underset{\sim}{r}_\gamma , \tag{17a}$$

$$\underset{\sim}{n}_\eta = -b_{2\beta} g^{\beta\gamma} \underset{\sim}{r}_\gamma \quad . \tag{17b}$$

If now g_{11}, g_{12}, g_{22}, b_{11}, b_{12}, b_{22} are arbitrarily prescribed then the set of Eqs. (14) and (17), which represent fifteen scalar equations for the nine scalars ($\underset{\sim}{r}_\xi$, $\underset{\sim}{r}_\eta$, $\underset{\sim}{n}$), form an overdetermined system. Consequently one has to impose the compatability requirements

$$(\underset{\sim}{r}_{\alpha\beta})_\gamma = (\underset{\sim}{r}_{\alpha\gamma})_\beta$$

for all values of α, β, γ from 1 to 2. This operation leads to the Mainardi-Codazzi equations and the theorema egregium of Gauss which are higher order equations and are not very suitable for the purpose of numerical solution. We therefore return to the Gauss equations (14) and ask the question: Is it possible to develop a method which centers around the Gauss equations and is simple to implement numerically? The answer is in affirmative if we manipulate Eqs. (14) as follows.

Multiplying Eq. (14a) by g_{22}, Eq. (14b) by $-2g_{12}$ and Eq. (14c) by g_{11} and adding the three equations, we get

$$\begin{aligned} \mathcal{L}\underset{\sim}{r} = &-[(\Delta_2\xi)\underset{\sim}{r}_\xi + (\Delta_2\eta)\underset{\sim}{r}_\eta]\, G_3 \\ &+ (g_{22}S - 2g_{12}T + g_{11}U)\underset{\sim}{n} \ , \end{aligned} \tag{18}$$

where $\mathcal{L}$ is the second order differential operator,

$$\mathcal{L} = g_{22}\partial_{\xi\xi} - 2g_{12}\partial_{\xi\eta} + g_{11}\partial_{\eta\eta} \ ,$$

and Δ_2 is the second order differential operator of Beltrami. For any surface x^ν = const., (refer to the scheme in (11)),

$$\begin{aligned} \Delta_2^{(\nu)} = &\frac{1}{\sqrt{G_\nu}}\,[\partial_\alpha\{\frac{1}{\sqrt{G_\nu}}(g_{\beta\beta}\partial_\alpha - g_{\alpha\beta}\partial_\beta)\} \\ &+ \partial_\beta\{\frac{1}{\sqrt{G_\nu}}(g_{\alpha\alpha}\partial_\beta - g_{\alpha\beta}\partial_\alpha)\}] \quad . \end{aligned} \tag{19a}$$

In particular for the surface ζ = const. we drop the enclosed superscript and write

$$\Delta_2 = \frac{1}{\sqrt{G_3}}\left[\partial_\xi\left\{\frac{1}{\sqrt{G_3}}(g_{22}\partial_\xi - g_{12}\partial_\eta)\right\}\right.$$

$$\left. + \partial_\eta\left\{\frac{1}{\sqrt{G_3}}(g_{11}\partial_\eta - g_{12}\partial_\xi)\right\}\right] . \qquad (19b)$$

It is easy to show by using the definitions of $T^\delta_{\infty\beta}$, that

$$\Delta_2\xi = \frac{1}{G_3}(2g_{12}T^1_{12} - g_{22}T^1_{11} - g_{11}T^1_{22}) , \qquad (20a)$$

$$\Delta_2\eta = \frac{1}{G_3}(2g_{12}T^2_{12} - g_{22}T^2_{11} - g_{11}T^2_{22}) . \qquad (20b)$$

The system of Eqs. (18) is still untamed and needs suitable constraints. We must also somehow modify the terms S, T, U so as to bring the variation of $\underset{\sim}{r}$ with respect to ζ, as was noted in the opening paragraph of this section. To achieve this objective we consider the Eqs. (8c) which for the surface ζ = const. are

$$\underset{\sim}{r}_{\xi\xi} = \Gamma^1_{11}\underset{\sim}{r}_\xi + \Gamma^2_{11}\underset{\sim}{r}_\eta + \Gamma^3_{11}\underset{\sim}{r}_\zeta , \qquad (21a)$$

$$\underset{\sim}{r}_{\xi\eta} = \Gamma^1_{12}\underset{\sim}{r}_\xi + \Gamma^2_{12}\underset{\sim}{r}_\eta + \Gamma^3_{12}\underset{\sim}{r}_\zeta , \qquad (21b)$$

$$\underset{\sim}{r}_{\eta\eta} = \Gamma^1_{22}\underset{\sim}{r}_\xi + \Gamma^2_{22}\underset{\sim}{r}_\eta + \Gamma^3_{22}\underset{\sim}{r}_\zeta , \qquad (21c)$$

where all the derivatives with respect to ζ are assumed to have been evaluated at ζ = const. Taking the dot product of Eqs. (21) with $\underset{\sim}{n}$ and comparing with Eqs. (13), we find that

$$\begin{aligned} b_{11} &= S = \lambda\Gamma^3_{11} , \\ b_{12} &= T = \lambda\Gamma^3_{12} , \qquad (22) \\ b_{22} &= U = \lambda\Gamma^3_{22} , \end{aligned}$$

where

$$\lambda = \underset{\sim}{n}\cdot\underset{\sim}{r}_\zeta = Xx_\zeta + Yy_\zeta + Zz_\zeta \quad , \tag{23}$$

$$X = (y_\xi z_\eta - y_\eta z_\xi)/\sqrt{G_3} \ ,$$

$$Y = (x_\eta z_\xi - x_\xi z_\eta)/\sqrt{G_3} \ , \tag{24}$$

$$Z = (x_\xi y_\eta - x_\eta y_\xi)/\sqrt{G_3} \ .$$

Thus, by using the forms in (22) we have established a connection with the coordinate ζ which changes from one surface to the next. We now rewrite Eq. (18) as

$$\mathcal{L}\underset{\sim}{r} + [(\Delta_2\xi)\underset{\sim}{r}_\xi + (\Delta_2\eta)\underset{\sim}{r}_\eta]G_3 = \underset{\sim}{n}R \ , \tag{25}$$

where

$$R = \lambda[g_{11}\Gamma^3_{22} - 2g_{12}\Gamma^3_{12} + g_{22}\Gamma^3_{11}) \ . \tag{26a}$$

Note that

$$R = G_3(k_1 + k_2) \tag{26b}$$

where $k_1 + k_2$ is twice the mean curvature of the surface.

§3.1 Fundamental generating system of equations

We now impose the following differential constraints on the coordinates ξ and η:

$$\Delta_2\xi = 0 \ , \tag{27a}$$

$$\Delta_2\eta = 0 \ , \tag{27b}$$

and take them as the fundamental generating equations for the coordinates in a surface. It must be noted that Δ_2 is not a 2D Laplace operator except when the surface degenerates into a plane having no dependence on z.

It is a well known result in differential geometry that the isothermic coordinates in a surface satisfy Eqs. (27) identically. The isothermic coordinates ξ and η are those orthogonal coordinates in a surface which yield $g_{22} = g_{11}$. The situation here is parallel to the choice of the Laplace equations $\nabla^2\xi = 0$, $\nabla^2\eta = 0$ for the generation of plane curvilinear coordinates, (e.g., the TTM method[2]), which are also satisfied identically by the conformal coordinates in a plane. This does not mean that the Laplace equations are suitable only for the generation of conformal coordinates. In fact, as is evidenced by the available body of numerical results, the Laplace equations are capable of generating very general coordinates in arbitrary domains. Therefore, there looks to be no apparent reason why Eqs. (27) should not form the basic generating system for general coordinates in a surface. The analytical solutions given in this paper and the numerical results given in Warsi and Ziebarth[15] support this contention.

Having chosen Eqs. (27) as the generating system, the equation for the determination of the Cartesian coordinates, viz., Eq. (25), becomes

$$\mathcal{L}\underset{\sim}{r} = n\underset{\sim}{R} \ . \tag{28}$$

The three scalar equations in expanded form are

$$g_{22}x_{\xi\xi} - 2g_{12}x_{\xi\eta} + g_{11}x_{\eta\eta} = XR \ , \tag{29a}$$

$$g_{22}y_{\xi\xi} - 2g_{12}y_{\xi\eta} + g_{11}y_{\eta\eta} = YR \ , \tag{29b}$$

$$g_{22}z_{\xi\xi} - 2g_{12}z_{\xi\eta} + g_{11}z_{\eta\eta} = ZR \ , \tag{29c}$$

where X, Y, Z, and R have been defined in Eqs. (24) and (26). It must be noted that by cyclic permutations, equations similar to Eqs. (29) can be written for the surfaces η = const. and ξ = const. However, only one set, e.g., Eqs. (29), is sufficient provided that we are able to take care of the derivatives $\underset{\sim}{r}_\zeta$ appearing in R.

The set of Eqs. (29) form a consistent set of equations for the determination of x, y, z under the prescribed boundary conditions.* For an analytical understanding of these equations we open the differentiations of the metric coefficients in the formulae for Γ^3_{11}, Γ^3_{12}, and Γ^3_{22}. Thus

$$\Gamma^3_{11} = \alpha x_{\xi\xi} + \beta y_{\xi\xi} + \gamma z_{\xi\xi} \ , \tag{30a}$$

$$\Gamma^3_{12} = \alpha x_{\xi\eta} + \beta y_{\xi\eta} + \gamma z_{\xi\eta} \ , \tag{30b}$$

*Refer to comment (i) at the end of the paper.

$$\Gamma^3_{22} = \alpha x_{\eta\eta} + \beta y_{\eta\eta} + \gamma z_{\eta\eta} , \qquad (30c)$$

where

$$\alpha = (G_5 x_\xi + G_6 x_\eta + G_3 x_\zeta)/g ,$$

$$\beta = (G_5 y_\xi + G_6 y_\eta + G_3 y_\zeta)/g ,$$

$$\gamma = (G_5 z_\xi + G_6 z_\eta + G_3 z_\zeta)/g .$$

Substituting Eqs. (30) in (26) and after arranging the terms we can rewrite Eqs. (29) as a quasilinear system,

$$A^{ij}_{\alpha\beta} \frac{\partial^2 x_j}{\partial x^\alpha \partial x^\beta} = 0 , \ i = 1, 2, 3 \qquad (31)$$

where $x_1 = x$, $x_2 = y$, $x_3 = z$ and there is an implicit sum on j from 1 to 3 and on α, β from 1 to 2. The coefficients $A^{ij}_{\alpha\beta}$ depend on the metric coefficients g_{11}, g_{12}, g_{22} and on those geometric quantities which depend only on the first partial derivatives. For example

$$A^{11}_{11} = g_{22}(1-\alpha\lambda X) , \ A^{11}_{12} = -2g_{12}(1-\alpha\lambda X) , \text{ etc., etc.}$$

Equations (31) are three equations in three unknowns with two independent variables. Refer to Petrovsky[16] for the classifications of such equations.

§3.2 Coordinate redistribution (concentration)

Before discussing the basic solution algorithm for the set of Eqs. (29) it is important to study the effect of a coordinate transformation which produces a nonuniform distribution of coordinates. Again using indexed quantities, let $\bar{x}^\alpha$ be another coordinate system defined as

$$\bar{x}^\alpha = \bar{x}^\alpha(x^1, x^2) , \ \alpha = 1, 2 ,$$

with

$$\det\left(\frac{\partial \bar{x}^\alpha}{\partial x^\beta}\right) \neq 0 .$$

Using x_i to mean either x, y, or z, we have

$$\frac{\partial x_i}{\partial x^\beta} = \frac{\partial x_i}{\partial \bar{x}^\gamma}\frac{\partial \bar{x}^\gamma}{\partial x^\beta}\ , \quad i = 1,\ 2,\ 3\ , \tag{32a}$$

$$\frac{\partial^2 x_i}{\partial x^\alpha \partial x^\beta} = \frac{\partial^2 x_i}{\partial \bar{x}^\delta \partial \bar{x}^\gamma}\frac{\partial \bar{x}^\delta}{\partial x^\alpha}\frac{\partial \bar{x}^\gamma}{\partial x^\beta} + \frac{\partial x_i}{\partial \bar{x}^\gamma}\frac{\partial^2 \bar{x}^\gamma}{\partial x^\alpha \partial x^\beta}\ . \tag{32b}$$

Also,

$$g^{\alpha\beta} = \bar{g}^{\theta\sigma}\frac{\partial x^\alpha}{\partial \bar{x}^\theta}\frac{\partial x^\beta}{\partial \bar{x}^\sigma}\ . \tag{32c}$$

Now, Eqs. (29) can be written in a compact form as

$$g^{\alpha\beta}\frac{\partial^2 x_i}{\partial x^\alpha \partial x^\beta} = \frac{R}{G_3} X_i\ , \tag{33}$$

where

$$X_1 = X\ ,\ X_2 = Y\ ,\ X_3 = Z\ ,$$

and

$$g^{11} = g_{22}/G_3\ ,\ g^{12} = -g_{12}/G_3\ ,\ g^{22} = g_{11}/G_3\ . \tag{34}$$

On coordinate transformation we have

$$G_3 = \bar{G}_3/(\overset{*}{D})^2\ ,\ R = \bar{R}/(\overset{*}{D})^2\ ,\ X_i = \bar{X}_i\ , \tag{35}$$

where

$$\overset{*}{D} = \frac{\partial x^1}{\partial \bar{x}^1}\frac{\partial x^2}{\partial \bar{x}^2} - \frac{\partial x^1}{\partial \bar{x}^2}\frac{\partial x^2}{\partial \bar{x}^1}\ .$$

Thus Eq. (33) becomes

$$\bar{g}^{\alpha\beta}\frac{\partial^2 x_i}{\partial \bar{x}^\alpha \partial \bar{x}^\beta} + \bar{g}^{\mu\sigma} P^\gamma_{\mu\sigma}\frac{\partial x_i}{\partial \bar{x}^\gamma} = \frac{\bar{R}}{\bar{G}_3}\bar{X}_i\ , \tag{36}$$

where

$$P^\gamma_{\mu\sigma} = \frac{\partial x^\alpha}{\partial \bar{x}^\mu}\frac{\partial x^\beta}{\partial \bar{x}^\sigma}\frac{\partial^2 \bar{x}^\gamma}{\partial x^\alpha \partial x^\beta}\ . \tag{37}$$

Using equations similar to (34) in the new coordinate system, Eq. (36) yields the equations

$$\bar{\mathcal{L}}x = \bar{X}\bar{R} \; , \tag{38a}$$

$$\bar{\mathcal{L}}y = \bar{Y}\bar{R} \; , \tag{38b}$$

$$\bar{\mathcal{L}}z = \bar{Z}\bar{R} \; , \tag{38c}$$

where

$$\bar{\mathcal{L}} = \bar{g}_{22}\partial_{\bar{\xi}\bar{\xi}} - 2\bar{g}_{12}\partial_{\bar{\xi}\bar{\eta}} + \bar{g}_{11}\partial_{\bar{\eta}\bar{\eta}} + \bar{P}\partial_{\bar{\xi}} + \bar{Q}\partial_{\bar{\eta}} \; , \tag{39}$$

$$\bar{P} = \bar{g}_{22}P^1_{11} - 2\bar{g}_{12}P^1_{12} + \bar{g}_{11}P^1_{22} \; , \tag{40a}$$

$$\bar{Q} = \bar{g}_{22}P^2_{11} - 2\bar{g}_{12}P^2_{12} + \bar{g}_{11}P^2_{22} \; , \tag{40b}$$

and $\bar{X}$, $\bar{Y}$, $\bar{Z}$, and $\bar{R}$ have exactly the same expressions as in (24) and (26a) in the new coordinate system.

The structure of the terms $P^{\gamma}_{\mu\sigma}$ is quite revealing particularly in those situations when it is desired to redistribute an already existing coordinate system x^{α} so as to achieve a desired concentration or expansion of the coordinates $\bar{x}^{\alpha}$. Though still a forcing function behavior for $P^{\gamma}_{\mu\sigma}$ has to be prescribed, the user is at least aware of its structure, that is, it must be composed of the product of two first partial derivatives and a second partial derivative. These considerations may be important in the adaptive coordinate systems. In other cases $P^{\gamma}_{\mu\sigma}$ may be prescribed arbitrarily. One such case has been treated numerically in Ref. 15. (Refer also to §3.)

§3.3 Morphology of A Solution Algorithm

The discussion that follows pertains to the case when it is desired to generate the 3D curvilinear coordinates between two artibrary shaped smooth surfaces. As is shown in Fig. 1, let the surface coordinates of the inner body $\eta = \eta_B$ and of the outer body $\eta = \eta_\infty$ be the same coordinates. Because of the right-handedness of the coordinate triple (ξ,η,ζ), the ordered pair (ζ,ξ) is taken as a positive ordered pair on both the surfaces. Since both the surfaces $\eta = \eta_B$ and $\eta = \eta_\infty$ are known either analytically or numerically, so that

$$\eta = \eta_B : \underset{\sim}{r} = \underset{\sim}{r}_B(\xi,\zeta) \; ; \; \eta = \eta_\infty : \underset{\sim}{r} = \underset{\sim}{r}_\infty(\xi,\zeta) \; , \tag{41}$$

and hence the needed partial derivatives with respect to ξ and ζ are directly available at the surface.

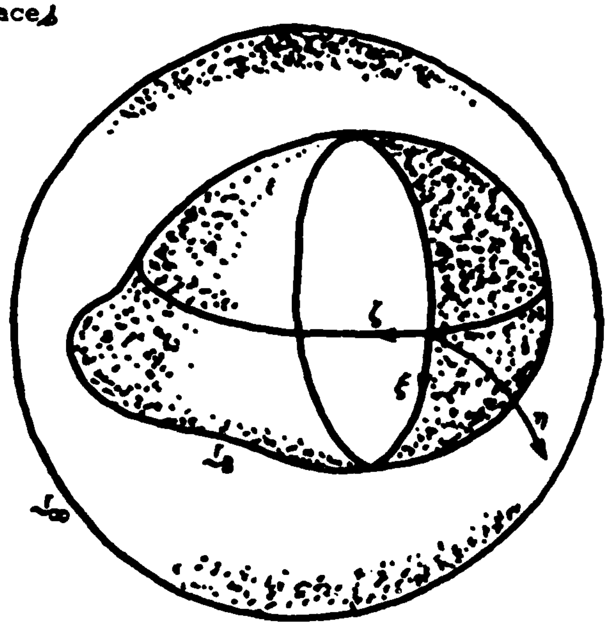

Figure 1. Selection of coordinates on the inner and outer boundaries.

For the computation of $\underset{\sim}{r}_\zeta$ in the field one must first note that the coordinate ζ may not, in general, satisfy the Beltrami's equation $\Delta_2^{(2)}\zeta = 0$. Consequently, $\underset{\sim}{r}_\zeta$ must satisfy the equation

$$\mathcal{L}^{(2)}\underset{\sim}{r} + G_2(\Delta_2^{(2)}\zeta)\underset{\sim}{r}_\zeta = G_2(k_1^{(2)} + k_2^{(2)})\underset{\sim}{n}^{(2)} \; .$$

From this equation we devise a weighted integral formula

$$\underset{\sim}{r}_\zeta = \int [f_1(\eta)(\underset{\sim}{r}_{\zeta\zeta})_B + f_2(\eta)(\underset{\sim}{r}_{\zeta\zeta})_\infty]d\zeta \; . \tag{42a}$$

where

$$(\underset{\sim}{r}_{\zeta\zeta})_{B,\infty} = [\frac{G_2}{g_{11}}(k_1^{(2)} + k_2^{(2)})\underset{\sim}{n}^{(2)} + \frac{2g_{13}}{g_{11}}\underset{\sim}{r}_{\xi\zeta} - \frac{g_{33}}{g_{11}}\underset{\sim}{r}_{\xi\xi}$$

$$- \frac{\sqrt{G_2}}{g_{11}}\{\frac{\partial}{\partial\zeta}(\frac{g_{11}}{\sqrt{G_2}}) - \frac{\partial}{\partial\xi}(\frac{g_{13}}{\sqrt{G_2}})\}\underset{\sim}{r}_\zeta]_{B,\infty} \; , \tag{42b}$$

and

$$f_1(\eta_B) = 1 \; , \; f_1(\eta_\infty) = 0 \; , \; f_2(\eta_B) = 0 \; , \; f_2(\eta_\infty) = 1 \; . \tag{42c}$$

Referring to Fig. 2(a), we now solve Eqs. (29) or (38) for each ζ = const., by prescribing the values of x, y and z on the lower curve C_1 and the upper curve C_2 which represent the curves on B and ∞ respectively. In Fig. 2(b) C_3 and C_4 are the cut lines on which periodic boundary conditions are to be imposed.

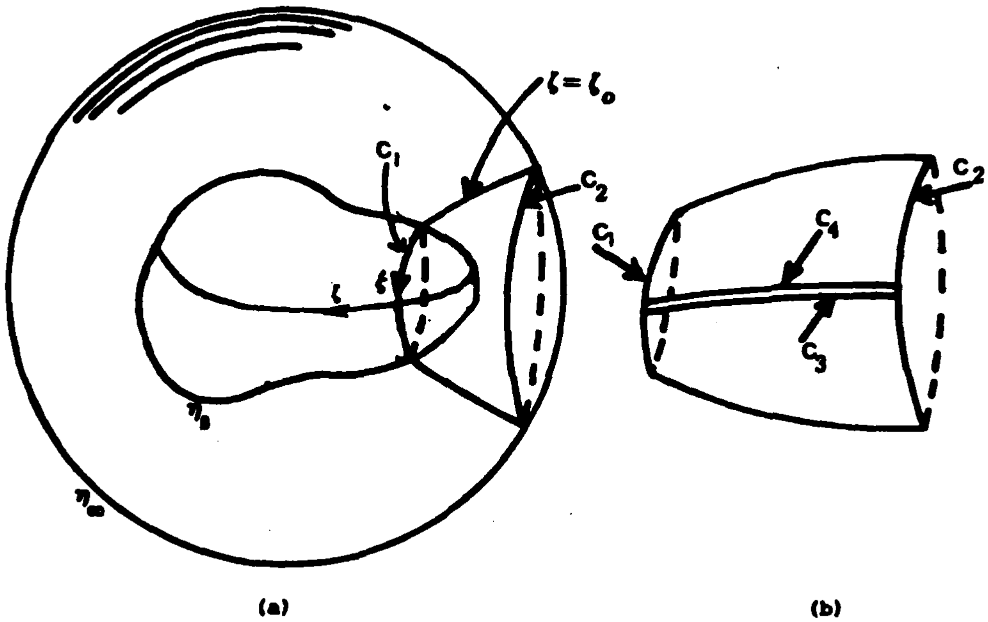

Figure 2.(a) Topology of the given surfaces. (b) Surface to be generated.

§3.4 Exact solutions

The following two examples demonstrate that the proposed set of generating equations (27) or equivalently the set of equations (29) or (38) are consistent and provide nontrivial solutions.

Example 1: Isothermic coordinates on a unit sphere.

Let the surface coordinates of a unit sphere be denoted as ξ, ζ, where the order (ζ,ξ) forms a right-handed system. Since our objective is to provide isothermic coordinates which are orthogonal, we assume

$$x = \psi(\zeta) \ , \ y = f(\zeta)\cos\ \xi \ , \ z = f(\zeta)\sin\ \xi \ , \tag{43a}$$

so that

$$f^2 + \psi^2 = 1 \ . \tag{43b}$$

Calculating the metric coefficients and the surface Christoffel symbols based on the assumed form (43a), we find that the equations $\overset{(2)}{\Delta}_2\xi = 0$ and $\overset{(2)}{\Delta}_2\zeta = 0$ are satisfied provided that

$$f^2 = \psi'^2 + f'^2 \ . \tag{43c}$$

Eliminating ψ between (43b,c), we get

$$f'^2 = (1-f^2)f^2 ,$$

which on integration yields

$$f(\zeta) = \frac{2e^{\zeta}}{1+e^{2\zeta}} , \ \psi(\zeta) = \frac{1-e^{2\zeta}}{1+e^{2\zeta}} . \tag{43d}$$

It can be verified that when the solution (43d) is used in (43a) then the resulting metric coefficients g_{11} and g_{33} are equal. Thus the coordinates ξ,ζ are isothermic. The relations between the standard spherical polar coordinates θ,ϕ and the coordinates ξ,ζ are

$$\xi = \phi \ , \ \zeta = \ell n \tan \frac{\theta}{2} \ .$$

Refer also to §5.1.1.

Example 2: <u>3D coordinates between a prolate ellipsoid and a sphere</u>.

We now consider the case of coordinate generation between an inner body $\eta = \eta_B$ which is a prolate ellipsoid and an outer body $\eta = \eta_\infty$ which is a sphere. The coordinates which vary on these two surfaces are ξ and ζ. A curve C_1 on the inner surface designated as $\zeta = \zeta_0$ is

$$x = \ell\cosh \eta_B \cos \zeta_0 \ , \ y = \ell\sinh \eta_B \sin \zeta_0 \cos \xi, \ z = \ell\sinh \eta_B \sin \zeta_0 \sin \xi. \tag{44a}$$

Similarly the curve C_2 corresponding to $\zeta = \zeta_0$ on the outer surface is

$$x = e^{\eta_\infty} \cos \zeta_0 \ , \ y = e^{\eta_\infty} \sin \zeta_0 \cos \xi \ , \ z = e^{\eta_\infty} \sin \zeta_0 \sin \xi \ . \tag{44b}$$

In order to provide the solution of the present problem with coordinate contraction, we consider Eqs. (38) and assume

$$\xi = \xi(\bar\xi) \ , \ \eta = \eta(\bar\eta) + \eta_B \tag{45}$$

where $\bar\xi = 0$ corresponds to $\xi = 0$ and $\bar\eta = \bar\eta_B$ corresponds to $\eta = \eta_B$. Thus $\xi(0) = 0$, $\eta(\bar\eta_B) = 0$. Under the transformation (45), the only nonzero components of $P^{\gamma}_{\mu\sigma}$ are P^1_{11} and P^2_{22}. Writing

$$\lambda(\bar\xi) = \frac{d\xi}{d\bar\xi} \ , \ \theta(\bar\eta) = \frac{d\eta}{d\bar\eta} \ .$$

we have

$$P^1_{11} = -\frac{1}{\lambda}\frac{d\lambda}{d\bar\xi} \ , \ P^2_{22} = -\frac{1}{\theta}\frac{d\theta}{d\bar\eta} \tag{46}$$

Based on the forms of the boundary conditions (44a) and (44b) we assume the following forms for x, y, z for $\zeta = \zeta_0$:

$$x = f(\bar{\eta})\cos\zeta_0 \ , \ y = \phi(\bar{\eta})\sin\zeta_0 \cos\xi, \ z = \phi(\bar{\eta})\sin\zeta_0 \sin\xi \ . \tag{47}$$

The boundary conditions for f and ϕ are*

$$f(\bar{\eta}_B) = \tau\cosh\eta_B \ , \ f(\bar{\eta}_\infty) = e^{\eta_\infty} \ , \ \phi(\bar{\eta}_B) = \tau\sinh\eta_B \ , \ \phi(\bar{\eta}_\infty) = e^{\eta_\infty} . \tag{48}$$

Using the expressions in (47) we calculate the various partial derivatives, metric coefficients, and all other data as needed for the Eqs. (38). On substitution we get an equation containing $\sin^2\zeta_0$ and $\cos^2\zeta_0$. Equating to zero the coefficients of $\sin^2\zeta_0$ and $\cos^2\zeta_0$ we obtain

$$\frac{f''}{f'} = \frac{\theta'}{\theta} + \frac{\phi'}{\phi}, \tag{49}$$

$$\frac{\phi''}{\phi'} = \frac{\theta'}{\theta} + \frac{\phi'}{\phi} \ , \tag{50}$$

where a prime denotes differentiation with respect to $\bar{\eta}$. On direct integrations of Eqs. (49) and (50) under the boundary conditions (48), we get

$$f(\bar{\eta}) = Ae^{B\eta(\bar{\eta})} + C \ ,$$

$$\phi(\bar{\eta}) = De^{B\eta(\bar{\eta})} \ ,$$

where

$$A = \tau[(e^{\eta_\infty} - \tau\cosh\eta_B)\sinh\eta_B]/(e^{\eta_\infty} - \tau\sinh\eta_B) \ ,$$

$$B = (\eta_\infty - \ln\tau\sinh\eta_B)/(\eta_\infty - \eta_B) \ ,$$

$$C = \tau[e^{\eta_\infty}(\cosh\eta_B - \sinh\eta_B)]/(e^{\eta_\infty} - \tau\sinh\eta_B) \ ,$$

$$D = \tau\sinh\eta_B \ .$$

As an application, we take

$$\xi(\bar{\xi}) = a\bar{\xi} \ , \ \eta(\bar{\eta}) = b(\bar{\eta} - \bar{\eta}_B)k^{\bar{\eta}} \ ,$$

*τ and η are the parameters of the ellipsoid.

where a, b and k are constants. Thus

$$\eta(\bar{\eta}) = \frac{(\eta_\infty - \eta_B)(\bar{\eta} - \bar{\eta}_B)}{\bar{\eta}_\infty - \bar{\eta}_B} k^{(\bar{\eta} - \bar{\eta}_\infty)} .$$

By taking a value of k slightly greater than one ($k = 1.05$) we can have sufficient contraction in the $\bar{\eta}$-coordinate near the inner surface. For the chosen problem since the dependence on ζ is simple, we find that the generated coordinates between a prolate ellipsoid and a sphere are

$$x = [Ae^{B\eta(\bar{\eta})} + C]\cos\zeta ,\ y = De^{B\eta(\bar{\eta})}\sin\zeta\cos\xi ,\ z = De^{B\eta(\bar{\eta})}\sin\zeta\sin\xi$$

This example shows that the chosen generating system of equations (38) are capable of providing non-isothermic coordinates between a prolate ellipsoid and a sphere.

§4. GENERATING DIFFERENTIAL EQUATIONS BASED ON LAPLACE EQUATIONS

For the purpose of coordinate generation in either two or three dimensions it has become quite popular, particularly after the publication of the TTM method[2], to adopt a system of inhomogeneous Laplace' equations as the generating system. The inhomogeneous terms are completely arbitrary and seemingly there is no guidance from the analytical side as to how they should be chosen. Because of this and due to other basic reasons it is important to reconsider the formulation of the problem of coordinate generation based on Laplace' system of equations from an analytical point of view. The conclusions drawn from these considerations are that the set of Laplace equations

$$\nabla^2 x^i = 0 ,\ i = 1, 2, 3 \tag{51}$$

are essentially the basis of the TTM method rather than the set of inhomogeneous equations

$$\nabla^2 \bar{x}^i = P^i(\bar{x}^1, \bar{x}^2, \bar{x}^3) ,\ i = 1, 2, 3 , \tag{52}$$

where P^i are the specified functions. The reason for this conclusion is that a coordinate transformation from x^i to any other system $\bar{x}^i$, both satisfying the same boundary conditions, automatically gives rise to the set of equations (52) from (51). Thus as soon as the solution of the system of equations (51) under the constraints of a body conforming boundary conditions has been obtained a transformation

$$\bar{x}^i = \bar{x}^i(x^1, x^2, x^3)$$

can redistribute these coordinates in any desired manner.

To formulate the above noted ideas analytically, we consider the formula for the Laplacian of a scalar ϕ in the curvilinear coordinate system,[12,13] which is

$$\nabla^2\phi = g^{ij}\left(\frac{\partial^2\phi}{\partial x^i \partial x^j} - \Gamma^r_{ij}\,\frac{\partial\phi}{\partial x^r}\right) . \tag{53}$$

If $\phi = x^m$ is any curvilinear coordinate, then from (53) we obtain

$$\nabla^2 x^m = -g^{ij}\,\Gamma^m_{ij} . \tag{54}$$

If $\phi = x_m$, where x_m is any of the rectangular Cartesian coordinate, $x_1 = x$, $x_2 = y$, $x_3 = z$, then since $\nabla^2 x_m = 0$, we obtain using (53),

$$g^{ij}\,\frac{\partial^2 x_m}{\partial x^i \partial x^j} + (\nabla^2 x^r)\,\frac{\partial x_m}{\partial x^r} = 0 . \tag{55}$$

Taking (51) as the basic generating system, we get from (55),

$$g^{ij}\,\frac{\partial^2 x_m}{\partial x^i \partial x^j} = 0 . \tag{56}$$

Using the formulae stated in §2, we get $\mathbf{D}x_m = 0$, or

$$\mathbf{D}x = 0 , \tag{57}$$

$$\mathbf{D}y = 0 , \tag{58}$$

$$\mathbf{D}z = 0 , \tag{59}$$

where the operator $\mathbf{D}$ is given by

$$\mathbf{D} = G_1\partial_{\xi\xi} + G_2\partial_{\eta\eta} + G_3\partial_{\zeta\zeta} + 2G_4\partial_{\xi\eta} + 2G_5\partial_{\xi\zeta} + 2G_6\partial_{\eta\zeta} . \tag{60}$$

In two dimensions* $g_{33} = 1$, $\frac{\partial}{\partial\zeta} = 0$, so that D becomes

$$D = g_{22}\partial_{\xi\xi} - 2g_{12}\partial_{\xi\eta} + g_{11}\partial_{\eta\eta} \ . \tag{61}$$

Let $\bar{x}^i$ be another coordinate system which satisfies the same body conforming boundary conditions as the system x^i, and let

$$\bar{x}^i = \bar{x}^i(x^1, x^2, x^3) \ , \ i = 1, 2, 3 \ ,$$

with

$$\det\left(\frac{\partial\bar{x}^i}{\partial x^j}\right) \neq 0 \ .$$

Then an analysis similar to §3.2 shows that

$$\frac{\partial x_m}{\partial x^j} = \frac{\partial x_m}{\partial\bar{x}^\ell}\frac{\partial\bar{x}^\ell}{\partial x^j} \ ,$$

$$\frac{\partial^2 x_m}{\partial x^i \partial x^j} = \frac{\partial^2 x_m}{\partial\bar{x}^k\partial\bar{x}^\ell}\frac{\partial\bar{x}^k}{\partial x^i}\frac{\partial\bar{x}^\ell}{\partial x^j} + \frac{\partial x_m}{\partial\bar{x}^\ell}\frac{\partial^2\bar{x}^\ell}{\partial x^i\partial x^j} \ .$$

Using the last expression and the transformation law

$$g^{ij} = \bar{g}^{rn}\frac{\partial x^i}{\partial\bar{x}^r}\frac{\partial x^j}{\partial\bar{x}^n}$$

in Eq. (56) we get

$$\bar{g}^{k\ell}\frac{\partial^2 x_m}{\partial\bar{x}^k\partial\bar{x}^\ell} + \bar{g}^{rn}P^\ell_{rn}\frac{\partial x_m}{\partial\bar{x}^\ell} = 0 \ , \tag{62}$$

where

$$P^\ell_{rn} = \frac{\partial x^i}{\partial\bar{x}^r}\frac{\partial x^j}{\partial\bar{x}^n}\frac{\partial^2\bar{x}^\ell}{\partial x^i\partial x^j} \ , \tag{63}$$

and is symmetric in the lower two indices. If now in Eq. (55) we replace x^i by $\bar{x}^i$, g^{ij} by $\bar{g}^{ij}$ and introduce

*Refer to comment (ii) at the end of the paper.

$$\nabla^2 \bar{x}^r = -\bar{g}^{mn} \bar{\Gamma}^r_{mn}$$

$$= P^r ,$$

then it amounts to the same thing as taking the non-homogeneous Laplace equations (52) as the generating system.† Thus we reach the conclusion that essentially Eqs. (51) are the basic generating equations and that any redistribution of the solution of Eqs. (51) gives rise to Eqs. (62).

Transferring the second term of Eq. (62) to the right hand side and using the formulae developed in §2 which are applicable to all coordinate systems, we obtain

$$\bar{D}x_m = -(G_1 P^{\ell}_{11} + G_2 P^{\ell}_{22} + G_3 P^{\ell}_{33} + 2G_4 P^{\ell}_{12} + 2G_5 P^{\ell}_{13} + 2G_6\, P^{\ell}_{23}) \frac{\partial x_m}{\partial \bar{x}^{\ell}} , \quad (64)$$

where $x_m = x$, y, or z, and $\bar{D}$ is the same operator as (60) in the new coordinate system. In two dimensions, Eq. (64) gives rise to the familiar forms[7,17]

$$\bar{D}x = -(\bar{g}_{22} P^1_{11} - 2g_{12} P^1_{12} + \bar{g}_{11} P^1_{22}) x_{\bar{\xi}} - (\bar{g}_{22} P^2_{11} - 2\bar{g}_{12} P^2_{12} + \bar{g}_{11} P^2_{22}) x_{\bar{\eta}} , \quad (65a)$$

$$\bar{D}y = -(\bar{g}_{22} P^1_{11} - 2\bar{g}_{12} P^1_{12} + \bar{g}_{11} P^1_{22}) y_{\bar{\xi}} - (\bar{g}_{22} P^2_{11} - 2\bar{g}_{12} P^2_{12} + \bar{g}_{11} P^2_{22}) y_{\bar{\eta}} . \quad (65b)$$

It must be noted that the preceding analysis guides one to a proper selection of the quantities P^{ℓ}_{rn} for concentrating the coordinate lines in the desired regions. This selection, though still arbitrary, at least suggests that the chosen P^{ℓ}_{rn} should be something like a product of two first and one second partial derivatives. This idea is important in the adaptive coordinate systems. Furthermore, the preceding analysis also exposes for the first time the existence of the cross derivative quantities P^{ℓ}_{rn} $(r \neq n)$ which do not appear if one starts from the Eqs. (52) and which may be important in non-orthogonal coordinates. For example, in two dimensions the quantities P^{ℓ}_{rn} are

$$P^1_{11} = (\xi_{\bar{\xi}})^2 \bar{\xi}_{\xi\xi} + 2\xi_{\bar{\xi}} \eta_{\bar{\xi}} \bar{\xi}_{\xi\eta} + (\eta_{\bar{\xi}})^2 \bar{\xi}_{\eta\eta} ,$$

$$P^2_{11} = (\xi_{\bar{\xi}})^2 \bar{\eta}_{\xi\xi} + 2\xi_{\bar{\xi}} \eta_{\bar{\xi}} \bar{\eta}_{\xi\eta} + (\eta_{\bar{\xi}})^2 \bar{\eta}_{\eta\eta} ,$$

†Refer to comment (iii) at the end of the paper.

$$P^1_{22} = (\xi_{\bar\eta})^2 \bar\xi_{\xi\xi} + 2\xi_{\bar\eta}\eta_{\bar\eta}\bar\xi_{\xi\eta} + (\eta_{\bar\eta})^2 \bar\xi_{\eta\eta} ,$$

$$P^2_{22} = (\xi_{\bar\eta})^2 \bar\eta_{\xi\xi} + 2\xi_{\bar\eta}\eta_{\bar\eta}\bar\eta_{\xi\eta} + (\eta_{\bar\eta})^2 \bar\eta_{\eta\eta} ,$$

$$P^1_{12} = \xi_{\bar\xi}\xi_{\bar\eta}\bar\xi_{\xi\xi} + (\xi_{\bar\xi}\eta_{\bar\eta} + \eta_{\bar\xi}\xi_{\bar\eta})\bar\xi_{\xi\eta} + \eta_{\bar\xi}\eta_{\bar\eta}\bar\xi_{\eta\eta} ,$$

$$P^2_{12} = \xi_{\bar\xi}\xi_{\bar\eta}\bar\eta_{\xi\xi} + (\xi_{\bar\xi}\eta_{\bar\eta} + \eta_{\bar\xi}\xi_{\bar\eta})\bar\eta_{\xi\eta} + \eta_{\bar\xi}\eta_{\bar\eta}\bar\eta_{\eta\eta} .$$

If $\xi = \xi(\bar\xi)$ and $\eta = \eta(\bar\eta)$, then writing

$$\lambda = \frac{d\xi}{d\bar\xi} , \quad \theta = \frac{d\eta}{d\bar\eta}$$

we get

$$P^1_{11} = -\frac{1}{\lambda}\frac{d\lambda}{d\bar\xi} , \ P^2_{11} = 0 , \ P^1_{22} = 0 ,$$

$$P^2_{22} = = \frac{1}{\theta}\frac{d\theta}{d\bar\eta} , \ P^1_{12} = 0 , \ P^2_{12} = 0 ,$$

which are exactly the same as have been used in an earlier paper.[17] In this case, writing for brevity

$$P^1_{11} = P , \ P^2_{22} = Q ,$$

Eqs. (65) simply become

$$\bar{D}x = -(\bar{g}_{22}Px_{\bar\xi} + \bar{g}_{11}Qx_{\bar\eta}) , \tag{66a}$$

$$\bar{D}y = -(\bar{g}_{22}Py_{\bar\xi} + \bar{g}_{11}Qy_{\bar\eta}) . \tag{66b}$$

These equations do not contain the cross derivative terms P^1_{12}, P^2_{12} because $\bar\xi$ and $\bar\eta$ have been chosen to be functions of ξ and η respectively.

§4.1 Case of orthogonal coordinates

In general, for the generation of orthogonal coordinates it is not necessary that the coordinate functions should also satisfy the Laplace equations in the xyz-space. In this section after summarizing the basic generating equations for the orthogonal coordinates we have studied the effect of constraining the coordinate functions to be simultaneously harmonic.

The orthogonality conditions are

$$g_{ij} = 0 \text{ for } i \neq j \ . \tag{67}$$

Also, for orthogonal coordinates Eqs. (54) simply become

$$\begin{aligned} \nabla^2 \xi &= \frac{1}{\sqrt{g}} \frac{\partial}{\partial \xi} \left(\frac{h_2 h_3}{h_1}\right) , \\ \nabla^2 \eta &= \frac{1}{\sqrt{g}} \frac{\partial}{\partial \eta} \left(\frac{h_1 h_3}{h_2}\right) , \\ \nabla^2 \zeta &= \frac{1}{\sqrt{g}} \frac{\partial}{\partial \zeta} \left(\frac{h_1 h_2}{h_3}\right) , \end{aligned} \tag{68}$$

where

$$\nabla^2 = \partial_{xx} + \partial_{yy} + \partial_{zz} \ , \ h_1 = \sqrt{g_{11}} \ , \ h_2 = \sqrt{g_{22}} \ , \ h_3 = \sqrt{g_{33}} \ , \ \sqrt{g} = h_1 h_2 h_3 \ .$$

Proceeding directly from Eq. (55) and using Eqs. (67) and (68) we obtain

$$\Xi x_m = 0 \ , \ m = 1, 2, 3 \ , \tag{69}$$

where

$$\Xi = \frac{\partial}{\partial \xi} \left(\frac{h_2 h_3}{h_1} \frac{\partial}{\partial \xi}\right) + \frac{\partial}{\partial \eta} \left(\frac{h_1 h_3}{h_2} \frac{\partial}{\partial \eta}\right) + \frac{\partial}{\partial \zeta} \left(\frac{h_1 h_2}{h_3} \frac{\partial}{\partial \zeta}\right) .$$

Note that the operator Ξ and the Laplacian operator ∇^2 are related as

$$\Xi \phi = h_1 h_2 h_3 \ \nabla^2 \phi \ ,$$

for a scalar ϕ.

Equations (69) are those fundamental equations which every orthogonal coordinate system must satisfy. A program of calculation using Eqs. (67) and (69) along with the definitions of g_{11}, g_{22} and g_{33} can be developed.

§4.1.1 Case of orthogonal coordinates using the Laplace equations

Case I: 3D coordinates.

If the generating system of equations is taken as

$$\nabla^2\xi = 0 \ , \ \nabla^2\eta = 0 \ , \ \nabla^2\zeta = 0 \ , \tag{70}$$

then from Eqs. (68) we find that

$$h_1 = f_2(\xi,\zeta)f_3(\xi,\eta) \ , \ h_2 = f_1(\eta,\zeta)f_3(\xi,\eta) \ , \ h_3 = f_1(\eta,\zeta)f_2(\xi,\zeta) \ , \tag{71}$$

where f_1, f_2, f_3 are arbitrary functions of their arguments. Also the generating system (69) for the Cartesian coordinates becomes

$$g_{22}g_{33}\frac{\partial^2 x_m}{\partial\xi^2} + g_{11}g_{33}\frac{\partial^2 x_m}{\partial\eta^2} + g_{11}g_{22}\frac{\partial^2 x_m}{\partial\zeta^2} = 0 \ , \ m = 1,\ 2,\ 3 \ , \tag{72}$$

which because of (71) can also be written as

$$f_1^2(\eta,\zeta)\frac{\partial^2 x_m}{\partial\xi^2} + f_2^2(\xi,\zeta)\frac{\partial^2 x_m}{\partial\eta^2} + f_3^2(\xi,\eta)\frac{\partial^2 x_m}{\partial\zeta^2} = 0 \ , \ m = 1,\ 2,\ 3 \ . \tag{73}$$

Case II: 2D coordinates.

For the case of 2D orthogonal coordinates the equations

$$\nabla^2\xi = 0 \ , \ \nabla^2\eta = 0 \ , \tag{74}$$

with the use of Eqs. (68) yield

$$g_{22} = ag_{11} \ ,$$

where a is a constant. The case a = 1 gives the corresponding isothermic coordinates which are conformal. However, by a straight forward coordinate transformation of the isothermic coordinates ξ,η to another coordinates $\bar{\xi},\bar{\eta}$ we can have a coordinate distribution in which $\bar{g}_{22} \neq \bar{g}_{11}$. For, let

$$\xi = \xi(\bar{\xi},\bar{\eta}) \ , \ \eta = \eta(\bar{\xi},\bar{\eta})$$

be an arbitrary orthogonal transformation. Using the chain rule of differentiation, we get

$$\xi_{xx} = \xi_{\bar{\xi}}\bar{\xi}_{xx} + \xi_{\bar{\eta}}\bar{\eta}_{xx} + \xi_{\bar{\xi}\bar{\xi}}(\bar{\xi}_x)^2 + \xi_{\bar{\eta}\bar{\eta}}(\bar{\eta}_x)^2 + 2\xi_{\bar{\xi}\bar{\eta}}\bar{\xi}_x\bar{\eta}_x$$

etc., etc.,

which when used in Eqs. (74) along with the orthogonality condition

$$\bar{\xi}_x\bar{\eta}_x + \bar{\xi}_y\bar{\eta}_y = 0$$

and the formulae

$$\nabla^2\bar{\xi} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial\bar{\xi}}\sqrt{\bar{g}_{22}/\bar{g}_{11}} \ , \qquad \text{(75a)}$$

$$\nabla^2\bar{\eta} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial\bar{\eta}}\sqrt{\bar{g}_{11}/\bar{g}_{22}} \ , \qquad \text{(75b)}$$

$$(\bar{\xi}_x)^2 + (\bar{\xi}_y)^2 = \frac{1}{\bar{g}_{11}} \ , \quad (\bar{\eta}_x)^2 + (\bar{\eta}_y)^2 = \frac{1}{\bar{g}_{22}} \ , \quad \bar{g} = \bar{g}_{11}\bar{g}_{22} \ ,$$

yield the equations

$$\sqrt{\bar{g}}\ \nabla^2\xi = \frac{\partial}{\partial\bar{\xi}}\left(\xi_{\bar{\xi}}\sqrt{\bar{g}_{22}/\bar{g}_{11}}\right) + \frac{\partial}{\partial\bar{\eta}}\left(\xi_{\bar{\eta}}\sqrt{\bar{g}_{11}/\bar{g}_{22}}\right) = 0 \ , \qquad \text{(76a)}$$

$$\sqrt{\bar{g}}\ \nabla^2\eta = \frac{\partial}{\partial\bar{\xi}}\left(\eta_{\bar{\xi}}\sqrt{\bar{g}_{22}/\bar{g}_{11}}\right) + \frac{\partial}{\partial\bar{\eta}}\left(\eta_{\bar{\eta}}\sqrt{\bar{g}_{11}/\bar{g}_{22}}\right) = 0 \ . \qquad \text{(76b)}$$

A study of Eqs. (76) suggests that if ξ is only a function of $\bar{\xi}$, and η is only a function of $\bar{\eta}$, e.g.,

$$\xi(\bar{\xi}) = \int \mu(\bar{\xi})d\bar{\xi} \ , \quad \eta(\bar{\eta}) = \int \frac{d\bar{\eta}}{\nu(\bar{\eta})} \ ,$$

then Eqs. (76a,b) are identically satisfied by taking

$$\sqrt{\bar{g}_{11}/\bar{g}_{22}} = \mu(\bar{\xi})\nu(\bar{\eta}) \ . \qquad \text{(76c)}$$

Thus

$$\bar{g}_{11} = \mu^2(\bar{\xi})\nu^2(\bar{\eta})\bar{g}_{22}, \qquad (76d)$$

and so the coordinates $\bar{\xi}, \bar{\eta}$ are orthogonal but not conformal.

An important result from the preceding analysis is that if the orthogonal coordinates are generated through the solution of the Laplace equations (74) then there exists an infinity of transformations $\xi = \xi(\bar{\xi})$, $\eta = \eta(\bar{\eta})$ in which the ratio $\bar{g}_{11}/\bar{g}_{22}$ is a product of a function of $\bar{\xi}$ and a function of $\bar{\eta}$. This result is not in general true for coordinates not satisfying the Laplace equations.

§5. GENERATING DIFFERENTIAL EQUATIONS BASED ON THE RIEMANN TENSOR

In any given space there are endless possibilities for the introduction of coordinate curves. Each chosen set of curves determines its own metric components. For example, in a Cartesian plane besides introducing rectangular Cartesian coordinates x, y, we also have endless possibilities for introducing either orthogonal or nonorthogonal coordinate curves. However, as is well known, there is a basic differential constraint on the variations of g_{ij}'s irrespective of the coordinate system. Since the curvature of an Euclidean two-dimensional plane is identically zero, the basic differential constraint on the g_{ij}'s is

$$(G_3)^{-\frac{1}{2}} R_{1212} = \frac{\partial}{\partial\eta}\left(\frac{\sqrt{G_3}}{g_{11}}\Gamma^2_{11}\right) - \frac{\partial}{\partial\xi}\left(\frac{\sqrt{G_3}}{g_{11}}\Gamma^2_{12}\right) = 0 , \qquad (77)$$

where ξ, η are any arbitrary coordinate curves in the plane. Thus no matter which coordinate system is introduced in a plane, the corresponding matrics g_{ij} must satisfy Eq. (77). Equation (77) has also been used as the basic generating equation for the generation of orthogonal coordinates in a plane[18]. In general, the Riemann curvature tensor R_{rjnp} defined as,[12,13]

$$R_{rjnp} = \frac{1}{2}\left(\frac{\partial^2 g_{rp}}{\partial x^j \partial x^n} + \frac{\partial^2 g_{jn}}{\partial x^r \partial x^p} - \frac{\partial^2 g_{rn}}{\partial x^j \partial x^p} - \frac{\partial^2 g_{jp}}{\partial x^r \partial x^n}\right)$$

$$+ g^{ts}([jn,s]\,[rp,t] - [jp,s][rn,t]) \qquad (78)$$

defines the components of the curvature tensor of any general space. If the space is N-dimensional, then the number of components R_{rjnp} are given by

$$\frac{N^2}{12}(N^2-1) .$$

Thus for N = 2 there is one distinct surviving component stated in Eq. (77). However, for N = 3, it has six distinct components

$$R_{1212},\ R_{1313},\ R_{2323},\ R_{1213},\ R_{1232},\ R_{1323}\ .$$

If the 3D-space is Euclidean, then its curvature is zero, so that the six equations

$$\begin{aligned} &R_{1212} = 0\ ,\ R_{1313} = 0\ ,\ R_{2323} = 0\ , \\ &R_{1213} = 0\ ,\ R_{1232} = 0\ ,\ R_{1323} = 0 \end{aligned} \tag{79}$$

determine the differential constraints for the six metric coefficients g_{ij} in any coordinate system introduced in an Euclidean space. These equations in the expanded form are as follows:

$$R_{1212} = \frac{\partial^2 g_{11}}{\partial\eta^2} - 2\,\frac{\partial^2 g_{12}}{\partial\xi\partial\eta} + \frac{\partial^2 g_{22}}{\partial\xi^2} + 2g^{ts}([22,s][11,t] - [12,s][12,t]) = 0\ , \tag{80a}$$

$$R_{1313} = \frac{\partial^2 g_{11}}{\partial\zeta^2} - 2\,\frac{\partial^2 g_{13}}{\partial\xi\partial\zeta} + \frac{\partial^2 g_{33}}{\partial\xi^2} + 2g^{ts}([33,s][11,t] - [13,s][13,t]) = 0\ , \tag{80b}$$

$$R_{2323} = \frac{\partial^2 g_{22}}{\partial\zeta^2} - 2\,\frac{\partial^2 g_{23}}{\partial\eta\partial\zeta} + \frac{\partial^2 g_{33}}{\partial\eta^2} + 2g^{ts}([33,s][22,t] - [23,s][23,t]) = 0\ , \tag{80c}$$

$$R_{1213} = \frac{\partial^2 g_{11}}{\partial\eta\partial\zeta} - \frac{\partial^2 g_{12}}{\partial\xi\partial\zeta} - \frac{\partial^2 g_{13}}{\partial\xi\partial\eta} + \frac{\partial^2 g_{23}}{\partial\xi^2} + 2g^{ts}([23,s][11,t] - [12,s][13,t]) = 0, \tag{80d}$$

$$R_{1232} = \frac{\partial^2 g_{22}}{\partial\xi\partial\zeta} - \frac{\partial^2 g_{12}}{\partial\eta\partial\zeta} - \frac{\partial^2 g_{23}}{\partial\xi\partial\eta} + \frac{\partial^2 g_{13}}{\partial\eta^2} + 2g^{ts}([22,s][13,t] - [23,s][12,t]) = 0, \tag{80e}$$

$$R_{1323} = \frac{\partial^2 g_{33}}{\partial\xi\partial\eta} - \frac{\partial^2 g_{13}}{\partial\eta\partial\zeta} - \frac{\partial^2 g_{23}}{\partial\xi\partial\zeta} + \frac{\partial^2 g_{12}}{\partial\zeta^2} + 2g^{ts}([33,s][12,t] - [23,s][13,t]) = 0, \tag{80f}$$

where $[ij,k]$ are the Christoffel symbols of the first kind defined in (8a).

Equations (80) are those consistent set of partial differential equations which must always be satisfied by the metric coefficients g_{ij}. In the 3D case Eqs. (80) are six equations in six unknowns and, therefore, they form a closed system of equations. In contrast, for the 2D case there is only one equation (Eq. (77)) and three unknowns g_{11}, g_{12}, g_{22} and therefore some constraints are needed to turn Eq. (77) (such as orthogonality[18]) into a solvable equation. This author is not aware of any numerical solution of the complete set of equations (80), though there are some possibilities of developing solution algorithms using Eqs. (80) as the core equations. For example, in the problem of obtaining the 3D coordinates for the configuration of Fig. 1, one can judiciously choose g_{11}, g_{13}, and g_{33} based on the given boundary data for the whole field and then solve Eqs. (80) for the remaining coefficients g_{22}, g_{23}, and g_{12}. It should also be noted that in any physical problem, e.g., the Navier-Stokes problem, one only needs the metric coefficients and their derivatives (Christoffel symbols), which become available after solving Eqs. (80). Nevertheless, for graphical and other purposes, one also needs the functions $x(\xi,\eta,\zeta)$ etc.

To obtain the Cartesian coordinates on the basis of the available g_{ij}'s, we introduce the unit base vectors $\underset{\sim}{\lambda}_i$ as

$$\underset{\sim}{\lambda}_i = \underset{\sim}{a}_i/\sqrt{g_{ii}} \text{ , no sum on i.} \tag{81}$$

Let the components of $\underset{\sim}{\lambda}_i$ along the rectangular Cartesian axes be denoted as u_i, v_i, w_i, so that

$$\underset{\sim}{\lambda}_i = (u_i, v_i, w_i) \text{ ,}$$

where

$$\begin{aligned} u_1 &= x_\xi/\sqrt{g_{11}} \text{ , } v_1 = y_\xi/\sqrt{g_{11}} \text{ , } w_1 = z_\xi/\sqrt{g_{11}} \text{ ,} \\ u_2 &= x_\eta/\sqrt{g_{22}} \text{ , } v_2 = y_\eta/\sqrt{g_{22}} \text{ , } w_2 = z_\eta/\sqrt{g_{22}} \text{ ,} \\ u_3 &= x_\zeta/\sqrt{g_{33}} \text{ , } v_3 = y_\zeta/\sqrt{g_{22}} \text{ , } w_3 = z_\zeta/\sqrt{g_{33}} \text{ .} \end{aligned} \tag{82}$$

Knowing u_i, v_i, w_i, it is possible to evaluate the Cartesian coordinates through the line integrals

$$\underset{\sim}{r} = \int (\underset{\sim}{\lambda}_1\sqrt{g_{11}}\, d\xi + \underset{\sim}{\lambda}_2\sqrt{g_{22}}\, d\eta + \underset{\sim}{\lambda}_3\sqrt{g_{33}}\, d\zeta) \text{ .} \tag{83}$$

The determination of u_i, v_i, w_i is a separate problem which we now consider. First of all using (81) in Eq. (8c), we get a system of first order partial differential equations

$$\frac{\partial \lambda_i}{\partial x^j} = \lambda_1 \left(\frac{g_{11}}{g_{ii}}\right)^{\frac{1}{2}} \Gamma^1_{ij} + \lambda_2 \left(\frac{g_{22}}{g_{ii}}\right)^{\frac{1}{2}} \Gamma^2_{ij}$$

$$+ \lambda_3 \left(\frac{g_{33}}{g_{ii}}\right)^{\frac{1}{2}} \Gamma^3_{ij} - \frac{\lambda_i}{2g_{ii}} \frac{\partial g_{ii}}{\partial x^j} , \tag{84}$$

where, as before, there is no sum on the repeated index i. Equations (84) form a system of 27 first order PDE's in nine independent variables u_i, v_i, w_i. This system of equations is overdetermined and thus its solvability should depend on certain compatibility conditions. According to a theorem on the overdetermined system of equations[19], if the compatibility conditions hold then the solution of Eqs. (84) exists and is unique. The conditions

$$\frac{\partial^2 \lambda_i}{\partial x^m \partial x^j} = \frac{\partial^2 \lambda_i}{\partial x^j \partial x^m} \tag{85}$$

for all values of i, m, and j are the compatibility conditions. To prove (85) we use Eq. (8c), which on cross differentiation yields

$$\frac{\partial^2 a_i}{\partial x^m \partial x^j} - \frac{\partial^2 a_i}{\partial x^j \partial x^m} = R^{\ell}_{.imj} a_{\ell} , \tag{86}$$

where $R^{\ell}_{.imj}$ is the Riemann-Christoffel curvature[12] tensor and is related with the Riemann's tensor $R_{ijk\ell}$. Evidently in our present case $R^{\ell}_{.imj} = 0$, since the space is Euclidean. Inserting (81) in (86) we find that Eq. (85) are identically satisfied.

It is interesting to note that for a two-dimensional curvilinear coordinate system there is no need to solve the system of equations such as (84). In this case the single differential equation with $G_3 = g$

$$R_{1212} = \sqrt{g}\left[\frac{\partial}{\partial \eta}\left(\frac{\sqrt{g}\,\Gamma^2_{11}}{g_{11}}\right) - \frac{\partial}{\partial \xi}\left(\frac{\sqrt{g}\,\Gamma^2_{12}}{g_{11}}\right)\right] = 0$$

implies the existence of a single function $\alpha(\xi,\eta)$ such that

$$\alpha_\xi = \frac{-\sqrt{g}}{g_{11}} \Gamma^2_{11} , \quad \alpha_\eta = \frac{-\sqrt{g}}{g_{11}} \Gamma^2_{12} .$$

Consequently

$$u_1 = \cos\alpha,\ v_1 = -\sin\alpha,\ u_2 = \cos(\alpha-\theta)\ ,\ v_2 = -\sin(\alpha-\theta)\ ,$$

where α is the angle made by the tanget to the coordinate line η = const. in a clockwise sense with the x-axis, and

$$\cos\theta = g_{12}/\sqrt{g_{11}g_{22}}$$

is known.

§5.1 Case of orthogonal coordinates

We again return to the case of 3D orthogonal coordinates. Refer also to §4.1. Under the constraint of orthogonality,

$$\begin{aligned} &g_{12} = g_{13} = g_{23} = 0 \ , \ [12,3] = [13,2] = [23,1] = 0 \ , \\ &\Gamma^3_{12} = \Gamma^2_{13} = \Gamma^1_{23} = 0 \ , \ g = g_{11}g_{22}g_{33} \ , \end{aligned} \tag{87}$$

the set of equations (80) reduce somewhat. They are

$$\frac{\partial}{\partial\xi}\left(\frac{1}{\sqrt{g_{11}g_{22}}}\frac{\partial g_{22}}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(\frac{1}{\sqrt{g_{11}g_{22}}}\frac{\partial g_{11}}{\partial\eta}\right) + \frac{1}{2g_{33}\sqrt{g_{11}g_{22}}}\frac{\partial g_{11}}{\partial\zeta}\frac{\partial g_{22}}{\partial\zeta} = 0 \ , \tag{88a}$$

$$\frac{\partial}{\partial\xi}\left(\frac{1}{\sqrt{g_{11}g_{33}}}\frac{\partial g_{33}}{\partial\xi}\right) + \frac{\partial}{\partial\zeta}\left(\frac{1}{\sqrt{g_{11}g_{33}}}\frac{\partial g_{11}}{\partial\zeta}\right) + \frac{1}{2g_{22}\sqrt{g_{11}g_{33}}}\frac{\partial g_{11}}{\partial\eta}\frac{\partial g_{33}}{\partial\eta} = 0 \ , \tag{88b}$$

$$\frac{\partial}{\partial\eta}\left(\frac{1}{\sqrt{g_{22}g_{33}}}\frac{\partial g_{33}}{\partial\eta}\right) + \frac{\partial}{\partial\zeta}\left(\frac{1}{\sqrt{g_{22}g_{33}}}\frac{\partial g_{22}}{\partial\zeta}\right) + \frac{1}{2g_{11}\sqrt{g_{22}g_{33}}}\frac{\partial g_{22}}{\partial\xi}\frac{\partial g_{33}}{\partial\xi} = 0 \ , \tag{88c}$$

$$\frac{\partial^2 g_{11}}{\partial\eta\partial\zeta} = \frac{1}{2}\frac{\partial g_{11}}{\partial\eta}\left(\frac{1}{g_{11}}\frac{\partial g_{11}}{\partial\zeta} + \frac{1}{g_{22}}\frac{\partial g_{22}}{\partial\zeta}\right) + \frac{1}{2g_{33}}\frac{\partial g_{11}}{\partial\zeta}\frac{\partial g_{33}}{\partial\eta} \ , \tag{88d}$$

$$\frac{\partial^2 g_{22}}{\partial\xi\partial\zeta} = \frac{1}{2}\frac{\partial g_{22}}{\partial\zeta}\left(\frac{1}{g_{22}}\frac{\partial g_{22}}{\partial\xi} + \frac{1}{g_{33}}\frac{\partial g_{33}}{\partial\xi}\right) + \frac{1}{2g_{11}}\frac{\partial g_{11}}{\partial\zeta}\frac{\partial g_{22}}{\partial\xi} \ , \tag{88e}$$

$$\frac{\partial^2 g_{33}}{\partial\xi\partial\eta} = \frac{1}{2}\frac{\partial g_{33}}{\partial\xi}\left(\frac{1}{g_{11}}\frac{\partial g_{11}}{\partial\eta} + \frac{1}{g_{33}}\frac{\partial g_{33}}{\partial\eta}\right) + \frac{1}{2g_{22}}\frac{\partial g_{22}}{\partial\xi}\frac{\partial g_{33}}{\partial\eta} \ , \tag{88f}$$

which are the Lame's equations.

§5.1.1 The case of isothermic coordinates.

Isothermic coordinates* in a surface embedded in a 3D Euclidean space are those coordinates in which the metric coefficients g_{11} and g_{33} in the surface η = const. are equal. That is, the element of length ds on η = const. is given by

$$(ds)^2_{\eta=\text{const.}} = g_{11}\,[(d\xi)^2 + (d\zeta)^2]\ ,$$

where ξ,ζ are chosen to be the surface coordinates. Setting

$$g_{33} = g_{11},\ \text{and}\ g_{22} = F(\eta)$$

in Eqs. (88), we obtain the basic equations for g_{11}, which are

$$\frac{\partial}{\partial\xi}\left(\frac{1}{g_{11}}\frac{\partial g_{11}}{\partial\xi}\right) + \frac{\partial}{\partial\zeta}\left(\frac{1}{g_{11}}\frac{\partial g_{11}}{\partial\zeta}\right) + \frac{1}{2Fg_{11}}\left(\frac{\partial g_{11}}{\partial\eta}\right)^2 = 0\ , \tag{89a}$$

$$\frac{\partial}{\partial\eta}\left(\frac{1}{\sqrt{Fg_{11}}}\frac{g_{11}}{\partial\eta}\right) = 0\ , \tag{89b}$$

$$\frac{\partial}{\partial\zeta}\left(\frac{1}{g_{11}}\frac{\partial g_{11}}{\partial\eta}\right) = 0\ , \tag{89c}$$

$$\frac{\partial}{\partial\xi}\left(\frac{1}{g_{11}}\frac{\partial g_{11}}{\partial\eta}\right) = 0\ . \tag{89d}$$

It can easily be verified that the only solution of Eqs. (89c,d) is

$$g_{11} = [a+P(\eta)]^2 f(\xi,\zeta)\ ,\ a = \text{const.} \tag{90}$$

Thus from (89b)

$$F(\eta) = \left(\frac{dP}{d\eta}\right)^2\ . \tag{91}$$

Substituting (90) and (91) in Eq. (89a), the differential equation for

*Refer to the comment (iv) at the end of the paper.

$f(\xi,\zeta)$ becomes

$$\frac{\partial}{\partial\xi}\left(\frac{1}{f}\frac{\partial f}{\partial\xi}\right) + \frac{\partial}{\partial\zeta}\left(\frac{1}{f}\frac{\partial f}{\partial\zeta}\right) + 2f = 0 \ . \tag{92}$$

In Kreyszig[14] we have the result that if in a portion of a surface isothermic coordinates can be introduced then that portion of the surface can conformally be mapped onto a plane. Thus in effect the solution of Eq. (92) provides that mapping function which conformally maps a surface onto a plane. As a verification of the above conclusion, we verify that the function

$$f = \frac{4e^{2\zeta}}{(1+e^{2\zeta})^2} \tag{93}$$

is a solution of Eq. (92). This function is related with the isothermic coordinates on a sphere. Using the parametric equation of a sphere

$$x = [a+P(\eta)] \cos\theta,\ y = [a+P(\eta)]\sin\theta \sin\phi,\ z = [a+P(\eta)]\sin\theta\cos\phi$$

and writing

$$\xi = \phi \ , \ \zeta = \ell n \tan\frac{\theta}{2} \ ,$$

where $0 < \phi < 2\pi$ and $0 < \theta < \pi$, we obtain

$$g_{33} = g_{11} = \frac{4(a+P)^2 e^{2\zeta}}{(1+e^{2\zeta})^2} \ .$$

Thus the equations

$$\begin{aligned} x &= \frac{(a+P)(1-e^{2\zeta})}{1+e^{2\zeta}} \ , \\ y &= \frac{2(a+P)e^{\zeta}\sin\xi}{1+e^{2\zeta}} \ , \\ z &= \frac{2(a+P)e^{\zeta}\cos\xi}{1+e^{2\zeta}} \end{aligned} \tag{94}$$

represent a sphere of radius $a+P(\eta)$ in terms of the isothermic coordinates ξ,ζ in the surface. Since $P(\eta)$ is an arbitrary function of η, we have the capability of prescribing a suitable function $P(\eta)$ to achieve any sort of contraction or expansion in the field. It looks that the representation (94) should prove useful in the computational problems associated with a sphere.

Comment (i):

As a further justification for the consistency of the set of Eqs. (29) it has been shown below that these equations can be combined to obtain the equation for a surface $z = z(x,y)$ in the well known form

$$\alpha z_{xx} - 2\beta z_{xy} + \gamma z_{yy} = 2HM\ , \tag{i}$$

where

$$2H = k_1+k_2 = R/G_3\ ,\ M = 1+p^2+q^2\ ,\ p = z_x,\ q = z_y\ ,$$

$$\alpha = (1+q^2)/\sqrt{M}\ ,\ \beta = pq/\sqrt{M}\ ,\ \gamma = (1+p^2)/\sqrt{M}\ .$$

First note the following definitions and identities:

$$\left.\begin{aligned}
&G_3 = g_{11}g_{22} - (g_{12})^2\ ,\ X = -p/\sqrt{M}\ ,\ Y = -q/\sqrt{M}\ ,\ Z = 1/\sqrt{M}\ ,\\
&\Delta_1(x,x) = (1-X^2)G_3\ ,\ \Delta_1(x,y) = -XYG_3\ ,\ \Delta_1(y,y) = (1-Y^2)G_3\ ,\\
&\text{where}\\
&\Delta_1(a,b) = g_{22}a_\xi b_\xi - g_{12}(a_\xi b_\eta + a_\eta b_\xi) + g_{11}a_\eta b_\eta\ .
\end{aligned}\right\} \tag{ii}$$

Calculating $z_{\xi\xi}$, $z_{\xi\eta}$, $z_{\eta\eta}$ from z_ξ, z_η, substituting these expressions in Eq. (29c) while using the equations in (ii) and Eqs. (29a,b) we recover Eq. (i) given above.

We now compare the equations obtained by Thomas[6] with those of Warsi[11]. Thomas' equations in the present notation are

$$\mathcal{L}x + 2pG_3H/\sqrt{M} = 0\ ,\ \mathcal{L}y + 2qG_3H/\sqrt{M} = 0\ ,\text{where } G_3 = (x_\xi y_\eta - x_\eta y_\xi)^2 M\ , \tag{iii}$$

which are exactly the same as Eqs. (29a,b) of this paper. It must, however, be pointed out that the derivation of Eqs. (iii) involves four steps:[6] (a) orthogonality of ζ with ξ,η, (b) vanishing of the principal curvature of ζ-lines, (c) elimination of an arbitrary parameter (which may be zero), (d) prescription of $z(x,y)$ for the surface to be generated. [redacted]

Comment (ii):

In two dimensions another differential system is provided by a first order Beltrami equation [20], which in the complex form is

$$f_{\bar{z}} - H(z,\bar{z})f_z = 0 \ , \tag{i}$$

where

$$f = f(z,\bar{z}) \ ,$$

$$z = x + iy \ , \ \bar{z} = x - iy \ , \ i = \sqrt{-1} \ .$$

Writing

$$f(z,\bar{z}) = \xi(x,y) + i\eta(x,y) \ ; \ H(z,\bar{z}) = \mu(x,y) + i\nu(x,y) \ , \tag{ii}$$

we obtain the following two real equations from (i):

$$-\eta_x = \beta\xi_x + \gamma\xi_y \ , \tag{iii}$$

$$\eta_y = \alpha\xi_x + \beta\xi_y \ , \tag{iv}$$

where

$$\alpha = [(1-\mu)^2 + \nu^2]/\Delta \ , \ \beta = -2\nu/\Delta \ , \ \gamma = [(1+\mu)^2 + \nu^2]/\Delta \ , \ \Delta = 1-(\mu^2 + \nu^2) \ .$$

Note that

$$\alpha\gamma - \beta^2 = 1 \ ,$$

$$\alpha + \gamma = 2(2-\Delta)/\Delta$$

A quasiconformal mapping becomes conformal when $H = 0$, or equivalently $\alpha = \gamma = 1$, $\beta = 0$. The resulting equations are then the Cauchy-Riemann equations

$$\xi_x = \eta_y \ , \ \xi_y = -\eta_x \ ,$$

and then $f(z)$ is an analytic function in the domain D.

Equations (iii) and (iv) can be inverted so that only the partial derivatives of x and y appear. Thus

$$x_\eta = \beta x_\xi - \alpha y_\xi \ , \tag{v}$$

$$y_\eta = \gamma x_\xi - \beta y_\xi \ . \tag{vi}$$

For Eqs. (v) and (vi) it is important to write α, β, γ in terms of the metric coefficients, which are[7,12],

$$\Delta = 4\sqrt{g}/[2\sqrt{g} + (g_{11} + g_{22})] \ ,$$

$$\alpha + \gamma = (g_{11} + g_{22})/\sqrt{g} \ ,$$

$$\alpha,\gamma = \left[g_{11} + g_{22} \mp \{(g_{11} + g_{22})^2 - 4(1 + \beta^2)g\}^{\frac{1}{2}}\right]/2\sqrt{g} \ .$$

Comment (iii):

As is expected, Eq. (82) can be reduced to the form

$$\bar{g}^{k\ell} \frac{\partial^2 x_m}{\partial \bar{x}^k \partial \bar{x}^\ell} + (\nabla^2 \bar{\xi}^r) \frac{\partial x_m}{\partial \bar{x}^r} = 0$$

by using the formula

$$\frac{\partial^2 \bar{x}^\ell}{\partial x^i \partial x^j} = \Gamma^p_{ij} \frac{\partial \bar{x}^\ell}{\partial x^p} - \bar{\Gamma}^\ell_{qr} \frac{\partial \bar{x}^q}{\partial x^i} \frac{\partial \bar{x}^r}{\partial x^j}$$

in the expression for $\dfrac{\partial^2 x_m}{\partial x^i \partial x^j}$. However, for gaining a new insight into the structure of the redistribution terms it looks profitable to keep the form (62) with P^ℓ_{rn} defined in (63).

Comment (iv):

Generation of isothermic coordinates can also be achieved by the method detailed in Ref. 14. Let $\tilde{x}^1$ and $\tilde{x}^2$ be some sort of coordinates introduced in a portion of the surface (for example from the subroutine developed by Craidon[21]), and let x^1, x^2 be the desired isothermic coordinates. Then

$$x^i = x^i(\tilde{x}^1,\tilde{x}^2) \ .$$

Because of x^i being isothermic, we have

$$g_{22} = g_{11} \ .$$

Using the transformation law for the covariant and the contravariant metric tensor components we get

$$\frac{\partial x^1}{\partial \tilde{x}^i} = \tilde{g}_{ij}\tilde{\varepsilon}^{jk}\frac{\partial x^2}{\partial \tilde{x}^k} , \tag{i}$$

$$\frac{\partial x^2}{\partial \tilde{x}^i} = -\tilde{g}_{ij}\tilde{\varepsilon}^{jk}\frac{\partial x^1}{\partial \tilde{x}^k} \tag{ii}$$

where

$$\tilde{\varepsilon}^{jk} = \frac{1}{\sqrt{\tilde{g}}} e_{jk} ,$$

$$e_{11} = 0 \ , \ e_{22} = 0 \ , \ e_{12} = +1 \ , \ e_{21} = -1 \ .$$

From Eqs. (i) and (ii) we find the second order differential equations

$$\frac{\partial}{\partial \tilde{x}^i}\left(\sqrt{\tilde{g}}\, \tilde{g}^{ij}\frac{\partial x^k}{\partial \tilde{x}^j}\right) = 0 , \tag{iii}$$

where $k = 1, 2$. Note that in the Eqs. (i) - (iii) the indices range over the values 1, 2.

Equations (iii) provide two linear uncoupled equations for the determination of the isothermic coordinates, since the values $\tilde{g}^{ij}$ of $\tilde{g}^{ij}$ are known a' priori.

CONCLUSIONS

Three distinct methods of numerical coordinate generation based on PDE's have been analyzed in detail. In the two newly proposed methods, viz., the methods discussed in §§3 and 5, some useful results have been obtained by looking at the generating system of equations as a system of forcing differential relations among the metric coefficients g_{ij}. For example, in the method of §3 and g_{ij}'s are forced to satisfy Eqs. (27) (refer also to their forms in Eqs. (20)). In the method of §5, the g_{ij}'s naturally satisfy Eqs. (80) since the space is intrinsically Euclidean. In the TTM method discussed in §4 the generating Laplace or Poisson equations also amount to specifying a set of differential constraints on the g_{ij}'s.

In the process of obtaining the above noted results a number of other results and equations have been obtained which should be satisfied by all coordinate systems. For example, the orthogonal coordinates in an Euclidean space must satisfy Eqs. (69), (88), and the nonorthogonal coordinates must satisfy Eqs. (80), no matter which method is used to generate them. In effect all these results provide enough material for proposing more efficient calculation algorithms for the coordinate generation on a computer.

ACKNOWLEDGMENTS

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NUMERICAL GENERATION OF THREE-DIMENSIONAL COORDINATES BETWEEN BODIES OF ARBITRARY SHAPES

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INTRODUCTION

This paper is devoted to the numerical solution of a set of second order elliptic partial differential equations for the generation of three-dimensional curvilinear coordinates between two arbitrary shaped bodies. The central idea of the method is to generate a series of surfaces between the given inner and the outer boundary surfaces and then to connect these surfaces in such a manner so as to have a sufficiently differentiable three-dimensional coordinate net in the enclosed region.

The basic analytical foundation of the present method has already been laid out by Warsi in §2 of Ref. 1. However, it is important to state here that the proposed equations for the numerical solution form a consistent set of second order elliptic equations which are a consequence of the equations of Gauss[2] for a surface. Additional constraints are then imposed which, besides yielding the simplest form of equations for numerical purposes, also preserve the essential geometric properties of the generated surfaces.

Formulation of the mathematical model

To fix ideas, let it be desired to generate the coordinates between the two surfaces designated as $\eta = \eta_B$ (the inner surface) and $\eta = \eta_\infty$ (the outer surface) respectively as is shown in Fig. 1 . The two coordinates which vary in these two surfaces are then labeled as ξ or I and ζ or K. The surfaces $\eta = \eta_B$ and $\eta = \eta_\infty$ are the known surfaces in which the Cartesian coordinates $\underset{\sim}{r} = (x,y,z)$ are given as functions of ξ and ζ, that is,

$$\underset{\sim}{r} = \underset{\sim}{r}_B(\xi,\zeta) \; , \; \underset{\sim}{r} = \underset{\sim}{r}_\infty(\xi,\zeta)$$

are known either numerically or analytically. The method to be discussed

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generates a surface for each fixed value of ζ or K starting from a curve on B and ending at the corresponding value of ζ or K on the outer boundary surface. Refer to Fig. 1b.

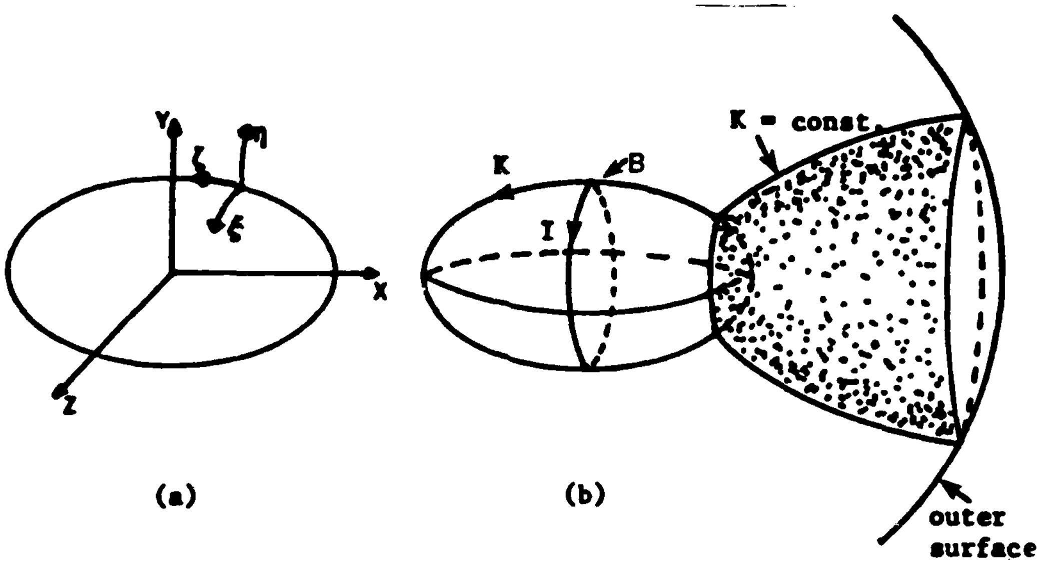

Figure 1(a). Coordinates ξ and ζ on the given surfaces. (b) the generated surface ζ = const.

Referring to Eq. (18) in Warsi[1], we now impose the restrictions

$$\Delta_2 \xi = \frac{1}{G_3}(2g_{12}T^1_{12} - g_{22}T^1_{11} - g_{11}T^1_{22}) = 0 \ , \tag{1}$$

$$\Delta_2 \eta = \frac{1}{G_3}(2g_{12}T^2_{12} - g_{22}T^2_{11} - g_{11}T^2_{22}) = 0 \ , \tag{2}$$

for ζ = const. In Eqs. (1) and (2) Δ_2 is the Beltrami's second order differential operator[1,2], and

$$g_{11} = x^2_\xi + y^2_\xi + z^2_\xi \ , \tag{3a}$$

$$g_{12} = x_\xi x_\eta + y_\xi y_\eta + z_\xi z_\eta \ , \tag{3b}$$

$$g_{22} = x^2_\eta + y^2_\eta + z^2_\eta \ , \tag{3c}$$

$$G_3 = g_{11}g_{22} - (g_{12})^2 \ , \tag{3d}$$

$$T^1_{11} = \frac{1}{2G_3}\left[g_{22}\frac{\partial g_{11}}{\partial \xi} + g_{12}\left(\frac{\partial g_{11}}{\partial \eta} - 2\frac{\partial g_{12}}{\partial \xi}\right)\right] \ , \tag{4a}$$

$$\Gamma^2_{22} = \frac{1}{2G_3}\left[g_{11}\frac{\partial g_{22}}{\partial \eta} + g_{12}\left(\frac{\partial g_{22}}{\partial \xi} - 2\frac{\partial g_{12}}{\partial \eta}\right)\right], \tag{4b}$$

$$\Gamma^1_{22} = \frac{1}{2G_3}\left[g_{22}\left(2\frac{\partial g_{12}}{\partial \eta} - \frac{\partial g_{22}}{\partial \xi}\right) - g_{12}\frac{\partial g_{22}}{\partial \eta}\right], \tag{4c}$$

$$\Gamma^2_{11} = \frac{1}{2G_3}\left[g_{11}\left(2\frac{\partial g_{12}}{\partial \xi} - \frac{\partial g_{11}}{\partial \eta}\right) - g_{12}\frac{\partial g_{11}}{\partial \xi}\right], \tag{4d}$$

$$\Gamma^1_{12} = \frac{1}{2G_3}\left(g_{22}\frac{\partial g_{11}}{\partial \eta} - g_{12}\frac{\partial g_{22}}{\partial \xi}\right), \tag{4e}$$

$$\Gamma^2_{12} = \frac{1}{2G_3}\left(g_{11}\frac{\partial g_{22}}{\partial \xi} - g_{12}\frac{\partial g_{11}}{\partial \eta}\right). \tag{4f}$$

Based on the structure of the Christoffel symbols $\Gamma^\alpha_{\beta\gamma}$ in Eqs. (4) we conclude that the constraining equations (1) and (2) are essentially a set of differential constraints on the variations of the metric coefficients $g_{\alpha\beta}$. Thus under the constraining equations (1) and (2), the three equations for the generation of the Cartesian coordinates x, y, z can be obtained. Below we write the equations when it is desired to have a concentration or expansion in the coordinates ξ and η, (refer to Eqs. (38) in Warsi[1]). For brevity of notation we use the same coordinates (ξ,η,ζ) either with or without coordinate redistributions. The equations are

$$\mathcal{L}x = XR, \tag{5a}$$

$$\mathcal{L}y = YR, \tag{5b}$$

$$\mathcal{L}z = ZR, \tag{5c}$$

where

$$\mathcal{L} = g_{22}\partial_{\xi\xi} - 2g_{12}\partial_{\xi\eta} + g_{11}\partial_{\eta\eta} + P\partial_\xi + Q\partial_\eta, \tag{6}$$

$$X = (y_\xi z_\eta - y_\eta z_\xi)/\sqrt{G_3}, \tag{7a}$$

$$Y = (x_\eta z_\xi - x_\xi z_\eta)/\sqrt{G_3}, \tag{7b}$$

$$Z = (x_\xi y_\eta - x_\eta y_\xi)/\sqrt{G_3}, \tag{7c}$$

$$R = (Xx_\zeta + Yy_\zeta + Zz_\zeta)(g_{11}\Gamma^3_{22} - 2g_{12}\Gamma^3_{12} + g_{22}\Gamma^3_{11}) \ , \tag{8}$$

$$\Gamma^3_{11} = \frac{1}{2g}[G_5 \frac{\partial g_{11}}{\partial \xi} + G_6(2 \frac{\partial g_{12}}{\partial \xi} - \frac{\partial g_{11}}{\partial \eta}) + G_3(2 \frac{\partial g_{13}}{\partial \xi} - \frac{\partial g_{11}}{\partial \zeta})] \ , \tag{9a}$$

$$\Gamma^3_{12} = \frac{1}{2g}[G_5 \frac{\partial g_{11}}{\partial \eta} + G_6 \frac{\partial g_{22}}{\partial \xi} + G_3(\frac{\partial g_{13}}{\partial \eta} + \frac{\partial g_{23}}{\partial \xi} - \frac{\partial g_{12}}{\partial \zeta})] \ , \tag{9b}$$

$$\Gamma^3_{22} = \frac{1}{2g}[G_5(2 \frac{\partial g_{12}}{\partial \eta} - \frac{\partial g_{22}}{\partial \xi}) + G_6 \frac{\partial g_{22}}{\partial \eta} + G_3(2 \frac{\partial g_{23}}{\partial \eta} - \frac{\partial g_{22}}{\partial \zeta})] \ , \tag{9c}$$

$$G_5 = g_{12}g_{23} - g_{13}g_{22} \ , \tag{10a}$$

$$G_6 = g_{12}g_{13} - g_{11}g_{23} \ , \tag{10b}$$

$$g = g_{33}G_3 + g_{13}G_5 + g_{23}G_6 \ , \tag{10c}$$

and G_3 has already been defined earlier in (3d).

A successful program of calculations based on the set of Eqs. (5) - (10) now rests on how effectively one can devise a calculation method for the first partial derivatives $\underset{\sim}{r}_\zeta = (x_\zeta, y_\zeta, z_\zeta)$ in the field. In this connection we first note that based on the prescribed values $\underset{\sim}{r}_B(\xi,\zeta)$, $\underset{\sim}{r}_\infty(\xi,\zeta)$ the partial derivatives with respect to ξ and ζ of any order can be evaluated on the given bodies. Thus we must somehow connect the evaluation of $\underset{\sim}{r}_\zeta$ in the field with the partial derivatives in the surface. To maintain the intrinsic geometrical properties of the ζ-lines in the field with the ζ-lines of the inner and the outer boundaries, we consider the differential equations for the surfaces η = const. Following the method in Warsi[1,3] we find that the coordinates ξ, ζ in any surface (including the given boundaries) must satisfy the equations

$$g_{33}\underset{\sim}{r}_{\xi\xi} - 2g_{13}\underset{\sim}{r}_{\xi\zeta} + g_{11}\underset{\sim}{r}_{\zeta\zeta} + (G_2\Delta_2^{(2)}\zeta)\underset{\sim}{r}_\zeta = G_2(k_1^{(2)}+k_2^{(2)})\underset{\sim}{n}^{(2)} \ , \tag{11}$$

where the enclosed superscript (2) in Eq. (11) means that all the quantities have been evaluated on the surface η = const. Also

$$G_2 = g_{11}g_{33} - (g_{13})^2 \ , \tag{12a}$$

$$G_2(k_1+k_2) = g_{33}U^{(2)} - 2g_{13}T^{(2)} + g_{11}S^{(2)} \ , \tag{12b}$$

$$U^{(2)} = \underset{\sim}{n}^{(2)} \cdot \underset{\sim}{r}_{\xi\xi} \ , \ T^{(2)} = \underset{\sim}{n}^{(2)} \cdot \underset{\sim}{r}_{\xi\zeta} \ , \ S^{(2)} = \underset{\sim}{n}^{(2)} \cdot \underset{\sim}{r}_{\zeta\zeta} \tag{13}$$

and

$$\underset{\sim}{n}^{(2)} = (X^{(2)}, Y^{(2)}, Z^{(2)}) \ ,$$

where

$$X^{(2)} = (y_\zeta z_\xi - y_\xi z_\zeta)/\sqrt{G_2} \ , \tag{14a}$$

$$Y^{(2)} = (x_\xi z_\zeta - x_\zeta z_\xi)/\sqrt{G_2} \ , \tag{14b}$$

$$Z^{(2)} = (x_\zeta y_\xi - x_\xi y_\zeta)/\sqrt{G_2} \ . \tag{14c}$$

It may be noted that $(k_1+k_2)/2$ is the mean curvature and S, T, U are the coefficients of the second fundamental form of the surface η = const.

Based on Eq. (11), we now formulate the following weighted integral formula for the evaluation of $\underset{\sim}{r}_\zeta$ in the field.

$$\underset{\sim}{r}_\zeta = \int [f_1(\eta)(\underset{\sim}{r}_{\zeta\zeta})_B + f_2(\eta)(\underset{\sim}{r}_{\zeta\zeta})_\infty] \, d\zeta \ , \tag{15a}$$

where

$$(\underset{\sim}{r}_{\zeta\zeta})_{B,\infty} = [\frac{G_2}{g_{11}} (k_1^{(2)}+k_2^{(2)})\underset{\sim}{n}^{(2)} + \frac{2g_{13}}{g_{11}} \underset{\sim}{r}_{\xi\zeta} - \frac{g_{33}}{g_{11}} \underset{\sim}{r}_{\xi\xi}$$

$$-(\frac{G_2}{g_{11}} \Delta_2^{(2)}\zeta)\underset{\sim}{r}_\zeta]_{B,\infty} \ , \tag{15b}$$

and

$$f_1(\eta_B) = 1 \ , \ f_1(\eta_\infty) = 0 \ , \ f_2(\eta_B) = 0 \ , \ f_2(\eta_\infty) = 1 \ . \tag{15c}$$

The functions $f_1(\eta)$ and $f_2(\eta)$ must satisfy the conditions (15c) and should be chosen to reflect the effect of the coordinate redistribution function Q appearing in Eq. (6). It is also to be noted that the coordinate ζ need not in general satisfy the Beltrami equation. That is, in general $\Delta_2^{(2)}\zeta \neq 0$.

Numerical solution of the equations

The numerical method used in this research for solving the system of Eqs. (5) is the method of finite difference using the point-SOR. First the coordinates ξ and ζ for both the inner surface ($\eta = \eta_B$) and the outer surface ($\eta = \eta_\infty$) are to be generated using the available x, y, z values for these surfaces either analytically or by a computer program developed by Craidon.[4] In this research we have used both the analytical methods where possible, and also the subroutine in Ref. 4 to generate the given body surface coordinates, with equal success. Three practical problems have to be resolved before an effective solution algorithm for Eqs. (5) can be developed. They are: (i) a specification of the functions $f_1(\eta)$ and $f_2(\eta)$ appearing in Eqs. (15), (ii) specification of the redistribution functions (concentration or expansion functions) P and Q, and (iii) a method to obtain the same coordinates on the inner and outer boundaries. We now discuss each problem in succession.

(i) Before discussing the specifications of $f_1(\eta)$ and $f_2(\eta)$ we may state that each value like $\eta = \eta_B$ and $\eta = \eta_\infty$ is a parameter to start with rather than an integer. The difference $\eta_\infty - \eta_B$ is the most important difference and is known as the "modulus of the domain." The determination of $\eta_\infty - \eta_B$ is a formidable problem in three dimensions but fortunately there is no need for it in the case of numerical coordinate generation. Writing

$$\bar{Z} = \frac{\eta_\infty - \eta}{\eta_\infty - \eta_B} , \tag{16a}$$

we find that the function f_1 defined in (15c) should be a function of $\bar{Z}$ only, so that

$$f_1(1) = 1 , \ f_1(0) = 0 , \tag{16b}$$

and

$$f_2(\bar{Z}) = 1 - f_1(\bar{Z}) . \tag{16c}$$

In the present computations we have taken f_1 and f_2 as linear functions of $\bar{Z}$, that is

$$f_1(\bar{Z}) = \bar{Z}. \tag{17a}$$

Other simple possibilities which have been tried are

Though for convex surfaces the method of spherical projection seems to be most desirable, we have for the present investigations, used the geometrical method of first surrounding the inner body by a sphere of diameter equal to the major length of the inner body. Next, each point (x, y_B, z_B) on the inner body is projected to a point (x, y_s, z_s) on the sphere surrounding the inner body. The correspondence between the inner and outer body is then made by extending a straight line from the center through (x, y_s, z_s) to a point $(x_\infty, y_\infty, z_\infty)$ on the outer sphere.

A number of program runs have been made for prolate ellipsoids of various thicknesses surrounded by sphere of large radii. Also a thin body of revolution with circular sections, resembling the fuselage of an airplane, surrounded by a sphere has been considered. These numerical results with and without coordinate concentration are shown in Figs. 2-7.

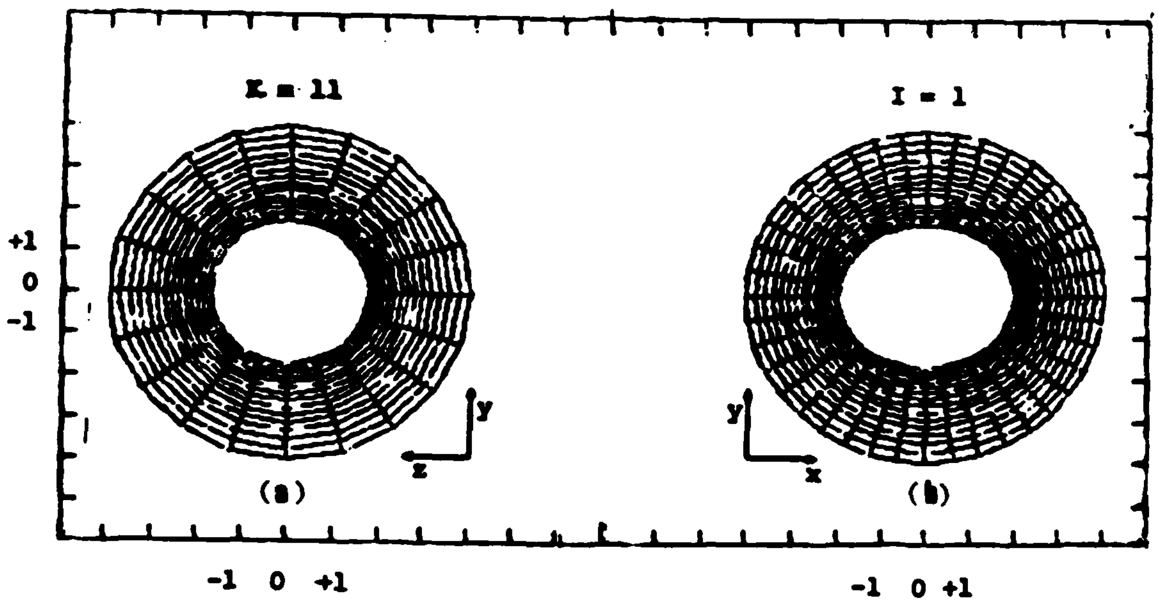

Figure 2. Inner body a thick prolate ellipsoid with major axis 2 and minor axis $\sqrt{3}$ surrounded by a sphere of radius 4. (a) Coordinate contours for a section ζ = const. (K = 11) for all (ξ,η) or (I,J) values, (b) for a section ξ = const. (I = 1) for all (η,ζ) or (J,K) values. In both cases no contraction in η, $\kappa = 1$.

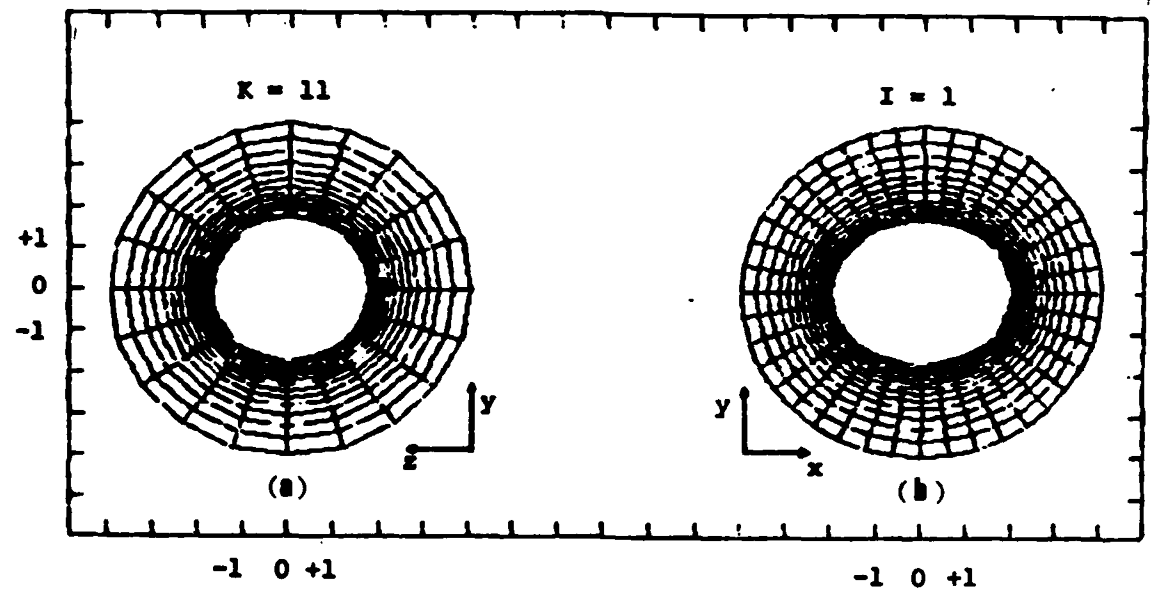

Figure 3. Cases (a) and (b) of Fig. 2, with contraction in η, κ = 1.05.

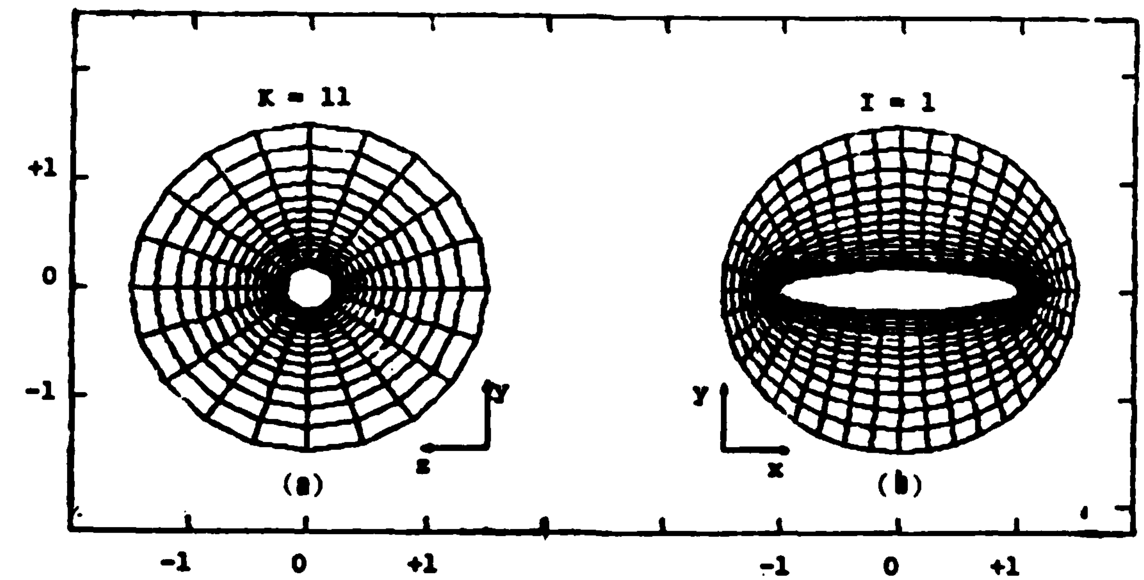

Figure 4. Inner body a thin prolate ellipsoid with major axis 1.02, minor axis 0.201 surrounded by a sphere of radius 1.5. (a) Coordinate contours for a section ζ = const. (K = 11) for all (ξ,η) or (I,J) values, (b) for a section ξ = const. (I = 1) for all (η,ζ) or (J,K) values. In both cases no contraction in η, κ = 1.

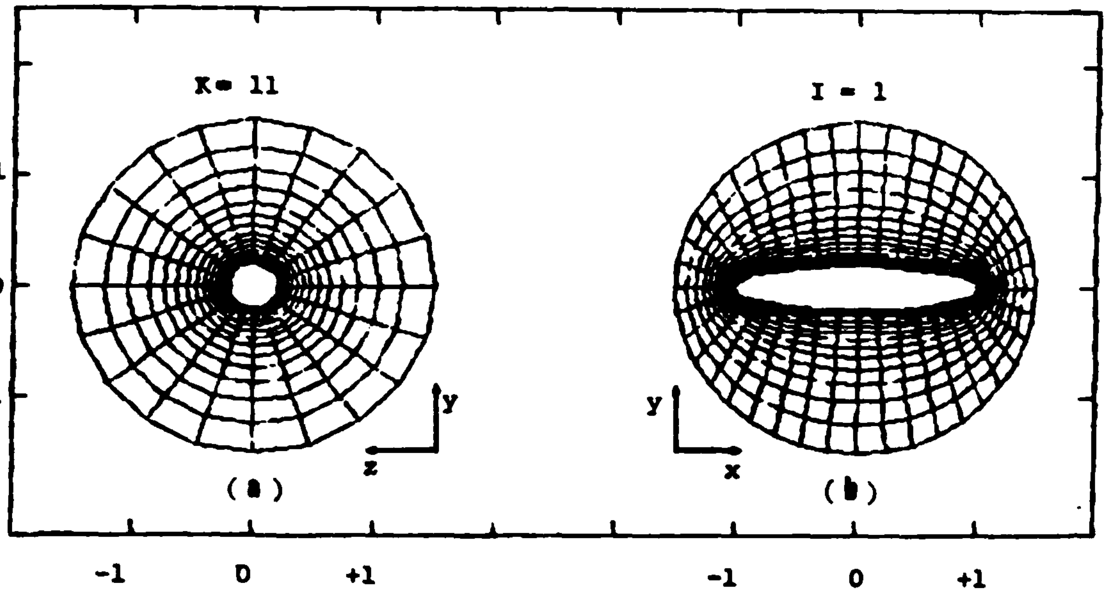

Figure 5. Cases (a) and (b) of Fig. 4, with contraction in η, $\kappa = 1.02$.

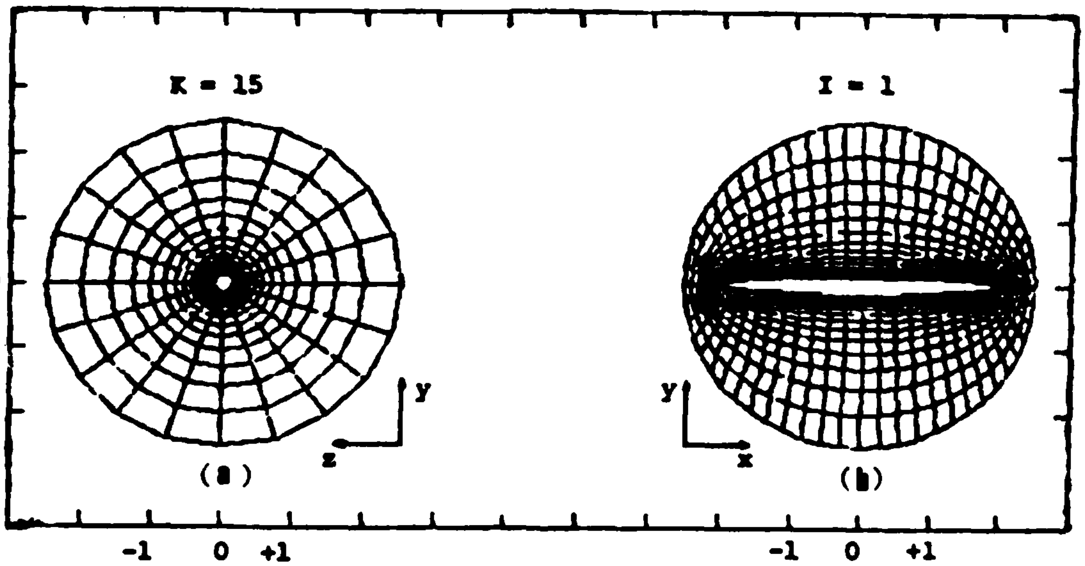

Figure 6. Inner body a thin body of revolution with circular sections having major axis 2 and minor axis 0.1313 surrounded by a sphere of radius 2.5. (a) Coordinate contours for a section ζ = const. (K = 15) for all (ξ,η) or (I,J) values, (b) for a section ξ = const. (I = 1) for all (η,ζ) or (J,K) values. In both cases no contraction in η, $\kappa = 1$.

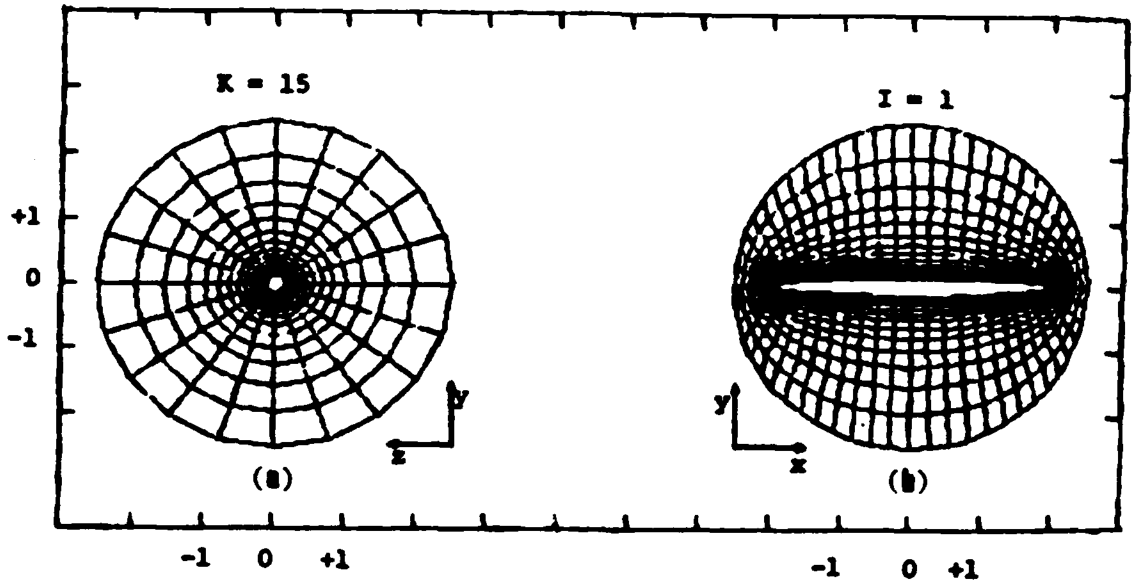

Figure 7. Cases (a) and (b) of Fig. 6, with contraction in η, $\kappa = 1.005$.

CONCLUSIONS

This paper has been devoted to the numerical solution of a set of elliptic equations for the purpose of numerically evolving a series of surfaces and the intersecting surfaces in arbitrary three-dimensional regions in space. The most difficult part of such a program is the generation of surfaces between any two given surfaces. This has been considered here for thick and thin prolate ellipsoids and a body of revolution forming the inner bodies and a sphere forming the outer boundary. Many successful numerical algorithms can be developed using the proposed equations as the core equations for providing the coordinates around a complete aircraft and other aerodynamical shapes.

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